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NICHOLSON'S MATHEMATICAL SERIES.

AN

ELEMENTARY ALGEBRA

THEORETICAL AND PRACTICAL

BY

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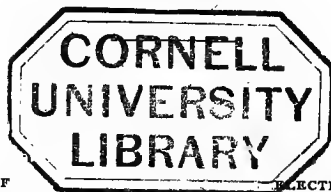
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PREFACE.

THIS work comprises about what is usually required for admission to the best colleges. In preparing it the aim has been—

1. To secure a clear and thorough treatment of the essential subjects in as compact a form as possible.

2. To present one difficulty at a time. For instance: (1) The several particulars in which Algebra is an extension of Arithmetic are introduced and treated successively, thus rendering easy and natural the passage of the student from the latter to the former. See articles 11, 69, 205, 323, 330. (2) Each subject is treated in a similar inductive manner, thereby enabling the student to progress with increasing interest, facility and intelligence. For example, see Simple Equations.

3. To embody a sufficient number of graded exercises for solution by the student in order to impress the principles on the memory, and to secure accuracy and rapidity in computation. These exercises were selected with great care; several of the examples being taken from mathematical journals of recent date, and many from the excellent works of Todhunter, Bland and other European writers.

The author feels assured that the simple and thorough manner in which the subject of Factoring is treated will meet the approval of teachers, as a thorough knowledge of this subject is essential to success in common algebraic work. Simple and Quadratic Equations have also received that broad and thorough consideration commensurate with their importance.

Originality of matter in a text-book of this kind is not to be expected, but it will be found that several subjects have been treated in a manner more or less original. The reader is

applications; Greatest Common Divisor, Fractions, Indeterminate and Impossible Problems, Inequalities, Variation, etc.

In conclusion, the work is earnestly commended to the use of instructors with the hope and belief that it will be found to be *teachable, practical and thorough*.

BATON ROUGE, LA.,

J. W. N.

August, 1888.

SUGGESTIONS TO TEACHERS.

1st. There are about 2500 examples in this book; while they are intended to furnish much practice, for it is only by practice of this kind that students become proficient in algebra, yet they are inserted rather to give the teacher a good choice from which to select than to require any student to solve them all.

2d. In passing over the book the first time, it is recommended to omit the more difficult examples and problems.

3d. Thoroughness can be attained only by frequent reviews and students should be drilled in factoring especially, until they are as familiar with it as they are with the multiplication table.

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ALGEBRA.

CHAPTER I.

POSITIVE AND NEGATIVE NUMBERS.

DEFINITIONS AND ELEMENTARY PRINCIPLES.

Art. 1. **Quantity** is anything which can be measured ; as distance, time, money.

2. To **measure** a quantity is to find how many times it contains another known quantity of the same kind, called the **Unit**.

Thus, to measure a certain distance, we would find how many times it contains the unit one yard; and to measure a quantity of corn, we would find how many times it contains the unit one bushel.

3. The object of measuring a quantity is to ascertain how much of it there is, and the result obtained is called the **Value** of the quantity.

4. The value of quantity is expressed by prefixing to the name of the unit the number which shows how many times that unit is contained in the quantity; as 4yd., 95gal.

5. A **Known Quantity** is one which has been measured, or one whose value is given; as \$7, 5ft., 14da., 11.

6. An **Unknown Quantity** is one which has not been measured, or one whose value is to be found.

7. A quantity is often measured by means of the relation it sustains to other known quantities. The process of thus measuring or determining a quantity is called Calculation or Deduction.

Thus, we may not know how much money D has, but if we know A has \$15, B \$10 more than A, C twice as much as B, and D as much as A, B and C together, we can measure or determine how much D has by calculating or deducing as follows:

As B has \$10 more than A he has \$25, as C has twice as much as B he has \$50, and as D has as much as A, B and C together he has \$90

8. The **Relative Values** of two or more quantities are the *numbers* which show how many times they contain a *common unit*.

Thus, if the lengths of three lines are 9 in., 11 in., and 15 in., their relative values are the numbers 9, 11, and 15, as these sustain the same relation to each other that the lines do.

9. The Relative Values of quantities are often called their **Numerical** or **Abstract Values**.

10. In Algebra, number and quantity are generally synonymous terms.

11. Algebra, like Arithmetic, treats of **numbers**, but it is an extension of Arithmetic in several particulars. The first is this:

Arithmetic treats only of positive numbers, while Algebra treats of both positive and negative numbers.

12. A **Positive Number** is one whose sign is plus (+), as +12, +7; and a **Negative Number** is one whose sign is minus (—), as —7, —50.

13. When no sign stands before a number the sign + is understood; thus, 7 is the same as + 7. But the sign — is never omitted.

14. Two numbers have **Like** signs when their signs are the same, as +3 and + 11, or — 6 and — 13; and they have **Unlike** signs when their signs are different, as 7 and — 5, or — 6 and + 15.

15. The **Absolute Value** of a quantity is its value without its sign. Thus, the absolute value of + 12 is 12, and of — 17, 17.

16. In Algebra, as in Arithmetic, the **Fundamental Operations** are **Addition, Subtraction, Multiplication, Division, Involution** and **Evolution**. But the introduction of negative numbers requires an extension of the meanings of some terms common to arithmetic and algebra. Now, since every such extension of meaning must be consistent with, and deducible from, its arithmetical signification, it is necessary to understand first the relation between positive and negative numbers.

17. As symbols of *nature* the signs + and — denote opposite qualities, conditions, motions, directions or tendencies.

It is immaterial which quality or direction we denote by the sign +, provided we denote its opposite by the sign —. Thus:

If gain is +, then loss is —; if distance or motion up is +, then distance or motion down is —; if warmer is +, then colder is —; if direction to the right is +, then direction to

the left is $-$; if time after a certain date is $+$, then time before that date is $-$; if time hence is $+$, then time since is $-$; if north latitude is $+$, then south latitude is $-$; if temperature above 0 is $+$, then temperature below 0 is $-$; if more is $+$, then less is $-$.

18. The Algebraic Value of a quantity is its value taken in connection with its sign of quality or direction.

Thus, the algebraic values of a gain of \$4 and a loss of \$7, are $+$ \$4 and $-$ \$7.

EXERCISE I.

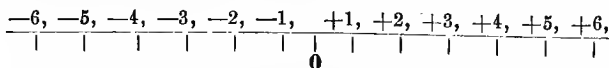
What are the algebraic values of the following:

1. A distance of 5 east and a distance of 8 west?
2. 65 days after now and 50 days before now?
3. 75 steps forward and 22 steps backward?
4. A credit of \$11 and a debit of \$13?
5. A fall of 3° in temperature and a rise of 5° ?
6. 17° north latitude and 15° south latitude?

NOTE.—Whatever $+$ may mean, $-$ means just the opposite.

7. If $+$ means east, read the following: A boy went $+$ 4 mi., then $-$ 3 mi., then $-$ 2 mi.
8. Read the same, calling $-$ south.
9. If $+$ means gain, read the following: A $+$ \$5, then $+$ \$4, then $-$ \$7, then $-$ \$1.

19. The relation of positive to negative numbers is shown in the following series:



The positive numbers begin at 0 and are laid off to the right; and the negative numbers begin at 0 and are laid off to the left.

ADDITION.

20. The **Sum** of two or more quantities is the result of combining them into one equivalent quantity.

21. **Addition** is the operation of finding the sum of two or more quantities.

22. The **Parts** are the quantities to be added.

23. The **Sign** of addition is +.

Quantities which are to be added or subtracted are often enclosed in parentheses in order that the signs + and —, which denote positive and negative numbers, may not be confounded with the signs + and — which indicate the operations of addition and subtraction.

24. The **Sign of Equality** is =, which is read *equals*, or *is equal to*.

ILLUSTRATIONS.

1. A gain of \$5 and a gain of \$3 are equivalent to a total gain of \$8.

Hence, + \$5 added to + \$3 = + \$8;

$$\text{or } (+ 5) + (+ 3) = + 8 \dots\dots (1)$$

2. A gain of \$5 and a loss of \$3 are equivalent to a net gain of \$2.

Hence, + \$5 added to — \$3 = + \$2;

$$\text{or } (+ 5) + (- 3) = + 2 \dots\dots (2)$$

3. A loss of \$5 and a gain of \$3 are equivalent to a net loss of \$2.

Hence, — \$5 added to + \$3 = — \$2;

$$\text{or } (- 5) + (+ 3) = - 2 \dots\dots (3)$$

4. A loss of \$5 and a loss of \$3 are equivalent to a total loss of \$8.

Hence, — \$5 added to — \$3 = — \$8;

$$\text{or } (- 5) + (- 3) = - 8 \dots\dots (4)$$

25. Hence, I. To add two numbers with like signs, as in (1) and (4), *add their absolute values, and prefix the common sign to the sum.*

II. To add two numbers with unlike signs, as in (2) and (3), *find the difference of their absolute values, and to it prefix the sign of the larger.*

26. Addition in Algebra does not always imply augmentation as it does in Arithmetic. For, when we add two numbers of opposite qualities they eliminate each other entirely or in part; and hence, the absolute value of the sum may be less than that of either of the parts. Such a result is called the *algebraic sum*, when it is necessary to distinguish it from the *arithmetical sum*. In Algebra, the terms *add* and *sum* should always be understood in an algebraic sense.

EXERCISE II.

Add together:

- | | |
|--------------------|------------------------|
| 1. -9 and $+3$. | 5. $+$ 7 and -5 . |
| 2. -7 and $+6$. | 6. $+$ 8 and -11 . |
| 3. $+5$ and $+9$. | 7. -6 and -4 . |
| 4. $+4$ and -4 . | 8. $+12$ and -7 . |

27. Since $(+8) + (+2)$ is the same as $+8+2$,
and $(+8) + (-2)$ “ “ “ “ $+8-2$,
we conclude:

1°. *The sum of two or more numbers may be indicated by writing them one after the other with their proper signs.*

2°. *A plus followed by a plus, $+(+)$, is equivalent to $+$, and a plus followed by a minus, as $+(-)$, is equivalent to $-$.*

Find the value of:

$$9. + 19 - 7.$$

$$12. 9 + + 3.$$

$$10. - 16 - 4.$$

$$13. 6 + - 4.$$

$$11. - 8 + 17.$$

$$14. 12 + - 15.$$

$$15. - 6 + 8 - 5 + 9.$$

$$1^{\circ}. \text{Solution: } - 6 + 8 = + 2; + 2 - 5 = - 3; \\ - 3 + 9 = + 6.$$

$$2^{\circ}. \text{Solution: } - 6 - 5 = - 11; + 8 + 9 = + 17; \\ - 11 + 17 = + 6.$$

$$16. + 10 + 3 - 7 + 2 - 8 - 9 + 4.$$

$$17. + 8 - 3 + 9 - 11 + 6 - 4 + 5.$$

$$18. - 6 + + 8 + - 7 + + 12 + - 4.$$

$$19. + 9 + 6 - 3 + + 8 + - 6 + - 3 - 5 + 7.$$

28. The Complementary Parts of a number are the numbers whose sum equals that number.

Thus, the complementary parts of $+ 4$ are $+ 3$ and $+ 1$, or $+ 7$ and $- 3$, or $- 5$, $+ 11$ and $- 2$.

Find the number whose complementary parts are:

$$20. - 6, + 4, + 9.$$

$$24. \$4, \$8, + \$3.$$

$$21. - 7, - 8, - 4.$$

$$25. - \$12, - \$9, + \$5.$$

$$22. + 9, + 8, - 3.$$

$$26. + 6 \text{ ft.}, + 7 \text{ ft.}, - 15 \text{ ft.}$$

$$23. + 11, + 6, - 5.$$

$$27. + + 3 \text{ mi.}, + - 12 \text{ mi.}$$

28. A boy made \$5, then spent \$3, then made \$11, then spent \$9; how much more did he then have than he had at first? $+ 5 - 3 + 11 - 9 = + 4$; hence, \$4 more.

29. A man paid \$8, then received \$9, then received \$7, then paid \$20; how much more did he then have than he had at first? Ans. $-\$12$.

30. B was born 5 yr. before A, C 6 yr. before B, D 7 yr. after C, and E 3 yr. after D; how much older is E than A?
31. B was born 7 yr. before A, C 5 yr. after B, D 8 yr. before C, and E 12 yr. after D; how much older is E than A?
32. A boy went 12 mi. east, then 9 mi. west, then 17 mi. west, then 15 mi. east; how far was he then, and in what direction, from the point of starting?
33. The temperature rose 8° , then fell 6° , then rose 5° , then fell 11° ; how much higher then was it than at first?
34. A ship is in latitude 20° north; if she sails 15° south, then 18° north, then 35° south; in what latitude will she then be?

29. A Parenthesis () is used to include within it such numbers as are to be considered as forming one number.

35. James has $(8 + 3)$ marbles and John $(7 - 4)$; how many have both?
 $(8 + 3) = 11$; $7 - 4 = 3$; $11 + 3 = 14$.
36. A ship sailed $(8 - 4 + 5)$ miles east, and then sailed $(6 + 7 - 5)$ miles west; how far was she then from the starting point?
37. Henry had $$(9 - 2 - 11)$, then spent $$(8 - 10 - 2)$, then earned $$(11 - 5 + 3)$; how many dollars did he then have?

SUBTRACTION.

30. Subtraction is the operation of finding the missing one of the complemental parts of a quantity.

31. The Subtrahend is the given part.

32. The Minuend is the quantity of which the subtrahend is a part.

33. The Difference is the required part.

34. The Sign of Subtraction is $-$, which means less, and is written before the subtrahend.

ILLUSTRATIONS.

1. If $+3$ is one part of $+5$, the other part is $+2$,
since $+3 + 2 = +5$. Hence,
 $+5$ less $+3 = +2$; or $(+5) - (+3) = +2$,
which is the same as $+5 - 3 = +2 \dots\dots (1)$
2. If -3 is one part of $+5$, the other part is $+8$,
since $-3 + 8 = +5$. Hence,
 $+5$ less $-3 = +8$; or $(+5) - (-3) = +8$,
which is the same as $+5 + 3 = +8 \dots\dots (2)$
3. If $+3$ is one part of -5 , the other part is -8 ,
since $+3 - 8 = -5$. Hence,
 -5 less $+3 = -8$; or $(-5) - (+3) = -8$,
which is the same as $-5 - 3 = -8 \dots\dots (3)$
4. If -3 is one part of -5 , the other part is -2 ,
since $-3 - 2 = -5$. Hence,
 -5 less $-3 = -2$; or $(-5) - (-3) = -2$,
which is the same as $-5 + 3 = -2 \dots\dots (4)$

35. From (1) and (3) it is evident that a minus followed by a plus, as $-(+)$, is equivalent to $-$; hence, *subtracting a positive number is equivalent to adding an equal negative number.*

36. From (2) and (4) it is evident that a minus followed by a minus, as $-(-)$, is equivalent to $+$; hence, *subtracting a negative number is equivalent to adding an equal positive number.*

37. Hence, to subtract one number from another, *change the sign of the subtrahend, or conceive it to be changed, and add it to the minuend.*

EXERCISE III.

1. One part of $+ 7$ is $.3$, what is the comp. part?
2. One part of $+ 9$ is $- 5$, " " " " "
3. One part of $- 11$ is $+ 6$, " " " " "
4. One part of $- 10$ is $- 12$, " " " " "
5. From $- 11$ take $- 4$.
6. From 9 take 13 .
7. From 9 take $- 13$.
8. From $- 7$ take $+ 3$.

Find the value of:

- | | |
|--------------------|-----------------------|
| 9. $+ 12 - (+ 6)$ | 12. $- 6 - (+ 8)$ |
| 10. $+ 13 - (- 7)$ | 13. $+ 8 - (- 9)$ |
| 11. $- 15 - (+ 8)$ | 14. $- (- 7) - (+ 3)$ |

How much more

15. Is 9 than 7?
16. Is 9 than $- 12$?
17. Is $- 9$ than $- 4$?
18. Is $- 14$ than $+ 6$?
19. Is $- 160$ than $- 345$?
20. Is $- 151$ than $+ 89$?

A man, after traveling two days, was 18 miles east of the point of starting; how far and in what direction did he go the second day:

21. If he went 10 mi. east the first day?
 $(+18 - (+10) = +8$, or 8 mi. east.)

22. If he went 20 mi. east the first day?
 $(+18 - (+20) = -2$, or 2 mi. west.)

23. If he went 7 mi. west the first day?

James and Henry have the same amount of money, how much more will James have than Henry:

24. If James gains \$121 and Henry gains \$76?
 $+\$121 - (+\$76) = +\$45$, or \$45 more.

25. If James gains \$94 and Henry loses \$17?

26. If James loses \$90 and Henry loses \$100?

27. If James loses \$225 and Henry gains \$148?

Moses and Joshua are together; how far will it be from Joshua to Moses:

28. If Moses goes 75 ft. east and Joshua 66 ft. east?
 $+75 \text{ ft.} - (+66 \text{ ft.}) = +9 \text{ ft.}$, or 9 ft. east.

29. If Moses goes 64 ft. east and Joshua 90 ft. east?

30. If Moses goes 25 ft. west and Joshua 46 ft. east? .

31. If Moses goes 37 ft. west and Joshua 72 ft. west?

38. When an expression within a parenthesis is preceded by the sign $-$, the parenthesis may be removed if the sign of every term within the parenthesis be changed. Thus:

$$(1) \ 8 - (6 + 3 - 4) = 8 - 6 - 3 + 4 = +3$$

$$(2) \ -(7 - 5 + 4) + (9 - 6 + 8) = -7 + 5 - 4 + 9 - 6 + 8 = +5$$

Find the value of:

$$32. 18 - 4 - (12 + 4 - 10 - 8)$$

$$33. 25 - (6 + 7) - (9 - 5 + 7 - 11).$$

$$34. -(16 - 4 + 7) + 10 - (8 + 13 - 4 + 1).$$

$$35. + (16 - 4 + 11 - 5) - (18 - 13 + 6 - 20).$$

39. When parentheses are inclosed within a parenthesis, they may be removed in succession by removing first, the innermost parenthesis; next, the innermost of all that remain, and so on. Thus:

$$\begin{aligned} & 9 - [8 + 6 - (7 - 4 + 5) - 11] \\ &= 9 - [8 + 6 - 7 + 4 - 5 - 11] \\ &= 9 - 8 - 6 + 7 - 4 + 5 + 11 = + 14. \end{aligned}$$

Find the value of:

$$36. 16 - [9 - (10 + 4 - 7) + 6] - 12.$$

$$37. 8 - (7 - 2) - [11 - (6 + 4 - 3)] + 6.$$

$$38. -[11 - \{8 - (6 - 8 + 7) + 6\} + 3] - 5.$$

$$39. 9 - (-8 + 7) - [13 - (7 - 6 + 15)]$$

$$40. 19 + 17 - (21 - 3) - [16 - \{11 - (10 - 4)\}]$$

MULTIPLICATION.

40. In arithmetic, multiplication is the operation of taking one number as many times as there are units in another.

41. The **Sign** of multiplication is $\times$.

42. According to the previous definition we have:

$$+ 3 \times + 5 = + (+ 5 + 5 + 5) = + (+ 15)$$

$$+ 3 \times - 5 = + (- 5 - 5 - 5) = + (- 15)$$

$$- 3 \times + 5 = - (+ 5 + 5 + 5) = - (+ 15)$$

$$- 3 \times - 5 = - (- 5 - 5 - 5) = - (- 15)$$

Hence (35, 36),

$$+ 3 \times + 5 = + 15 \dots (1) \quad - 3 \times + 5 = - 15 \dots (3).$$

$$+ 3 \times - 5 = - 15 \dots (2) \quad - 3 \times - 5 = + 15 \dots (4).$$

43. Hence, 1°. **Algebraic Multiplication** is the operation of taking one number as many times and in such a manner as is indicated by another.

2°. *Like signs produce +*, as in (1) and (4); *and unlike signs produce -*, as in (2) and (3).

44. The **Multiplicand** is the number to be multiplied.

45. The **Multiplier** is the number by which the multiplicand is multiplied.

46. The **Product** is the result of multiplication.

47. The **Factors** of a product are the multiplicand and multiplier.

EXERCISE IV.

Multiply:

1. $- 6$ by $+ 4$.

4. $+ 9$ by $+ 6$.

2. $+ 7$ by $- 3$.

5. $- 13$ by $- 34$.

3. $- 8$ by $- 7$.

6. $- 17$ by $+ 42$.

7. Find the value of $- 2 \times + 3 \times - 3 \times - 2 \times - 5$.

Solution: $- 2 \times + 3 = - 6$, $\times - 3 = + 18$, $\times - 2 = - 36$, $\times - 5 = + 180$.

8. Find the value of $+ 3 \times - 2 \times - 3 \times - 5 \times + 2$.

Solution: $+ 3 \times - 2 = - 6$, $\times - 3 = + 18$, $\times - 5 = - 90$, $\times + 2 = - 180$.

48. Hence, if an *even* number of factors have the sign —, the product will be +; and if an *odd* number of factors have the sign —, the product will be —.

Find the value of:

9. $+ 2 \times + 2 \times - 3$.
10. $+ 5 \times - 3 \times + 4 \times - 6$.
11. $- 6 \times - 5 \times - 4 \times + 3 \times - 2$.
12. $- 1 \times - 2 \times + 2 \times + 3 \times - 4 \times - 5$.
13. $(+ 5 + 4) \times (+ 8 - 11)$, or $(+ 9) \times (- 3)$.
14. $(6 - 8) \times (10 - 4)$.
15. $(7 - 5) \times (9 - 7) \times (5 - 9)$.
16. $(6 - 4 + 7) \times (3 - 8 + 12 - 20)$.

49. The **Complemental Factors** of a number are the numbers which when multiplied together will produce the given number.

Thus, the complemental factors of + 12 are + 4 and + 3, or — 4 and — 3, or + 2, — 2 and — 3.

50. Multiplication may be defined as the operation of finding a number when its complemental factors are given.

Find the number whose comp. factors are:

17. + 3, — 5, + 2, + 4.
18. + 2, — 3, — 4, — 5, + 6.
19. — 1, — 3, — 5, — 7, — 9.
20. $(6 - 2)$, $(7 - 8)$, $(9 - 3)$, $(4 - 14)$.
21. $+(6 - 2)$, $-(8 - 5)$, $-(3 - 8)$, $+(6 - 9 + 1)$.
22. $-(9 - 10)$, $-(7 + 1)$, $-(5 - 2)$, $-(7 - 18 + 4)$.

DIVISION.

51. Division is the operation of finding the missing one of the complemental factors of a number.

52. The Divisor is the given factor.

53. The Dividend is the number of which the divisor is a factor.

54. The Quotient is the required factor.

55. The Sign of division is $\div$, which is written before the divisor, and is read *divided by* or *contains*. Division is also indicated by writing the divisor under the dividend.

56. Since $+3 \times +4 = +12$, $+12 \div +3 = +4 \dots (1)$

Since $+3 \times -4 = -12$, $-12 \div +3 = -4 \dots (2)$

Since $-3 \times +4 = -12$, $-12 \div -3 = +4 \dots (3)$

Since $-3 \times -4 = +12$, $+12 \div -3 = -4 \dots (4)$

Hence, in division as in multiplication, *like signs produce +, and unlike signs -*.

EXERCISE V.

One factor:

1. Of $+30$ is $+5$, what is the comp. factor?
2. Of -35 is -7 , what is the comp. factor?
3. Of -54 is $+6$, what is the comp. factor?
4. Of $+63$ is -9 , what is the comp. factor?

What is the quotient:

5. If the dividend is -156 and the divisor $+13$?
6. If the dividend is -425 and the divisor $+17$?
7. If the dividend is $+784$ and the divisor -28 ?

8. Divide -40 by 8 .
9. Divide $+60$ by -5 .
10. Divide $(16 + 24)$ by $(17 - 9)$
11. Divide $(24 - 60 - 12)$ by $(11 - 17 - 6)$

How many times:

12. Is 4 contained in -120 ?
13. Is -3 contained in $+243$?
14. Is $(6 - 21)$ contained in $(7 - 112)$?
15. Is $-(8 - 2)$ contained in $-(5 - 341)$?

What is the value of:

16. $164 \div -8$?
17. $-38 \div +12$?
18. $-160 \div -25$?
19. $+110 \div -(17 - 2)$?
20. $(18 - 6 + 3) \div (9 - 4 + 5)$?
21. $(25 + 6 - 7) \div -(-12 + 9 - 3)$?
22. $(64 + 42 - 18) \div (+9 - 21 - 32)$

INVOLUTION.

57. A **Power** is the product of equal factors.

The *second power* or *square* is the product of *two* equal factors, and the *third power* or *cube* is the product of *three* equal factors. Thus, the second power or square of 4 is 4×4 , or 16 , and the third power or cube of 4 is $4 \times 4 \times 4$, or 64 .

58. **Involution** is the process of finding powers

59. The **Sign** of involution is an **Exponent** or **Index**, which is a small figure placed at the right of, and above, a number.

An exponent shows the required power of a number. Thus, 5^2 is the second power of 5 and is equal to 5×5 , or 25; 5^3 is the third power of 5 and is equal to $5 \times 5 \times 5$, or 125.

EXERCISE VI.

1. What is the fourth power of -3 ?

$$(-3)^4 = -3 \times -3 \times -3 \times -3 = +81.$$

2. What is the fifth power of -2 ?

$$(-2)^5 = -2 \times -2 \times -2 \times -2 \times -2 = -32.$$

60. Hence (48), an even power of a negative number is $+$, and an odd power of a negative number is $-$.

Find the value of:

3. $(-6)^3$

6. $(7 - 3)^4$

4. $(-7)^4$

7. $(3 - 11)^3$

5. $(+5)^6$

8. $(4 - 6 + 5)^5$

9. $5 \times 3^2 \times 2$

10. $7 \times (-2)^3 \times (-5)$

11. $4 \times (-5)^2 \times (-2)^3.$

Find the value of:

12. $2 \times (9 - 3)^2 \times (4 - 3)^5.$

13. $3 \times (2 - 3)^5 \times (5 - 4)^7.$

14. $7 \times (1 - 11)^3 \times (5 - 31 + 6)^2 \times (-1)^3.$

EVOLUTION.

61. A **Root** is one of the equal factors of a number.

The *Square Root* of a number is one of its *two* equal factors, and the *Cube Root* of a number is one of its *three* equal factors.

Thus, the square root of 64 is 8, since $8 \times 8 = 64$.

And the cube root of 64 is 4, since $4 \times 4 \times 4 = 64$.

62. **Evolution** is the process of finding roots,

63. The **Sign** of evolution is $\sqrt{}$, which is called the **Radical Sign**.

A figure, called the index, is written over the radical sign to show what root is to be taken. When no figure is written, 2 is understood. Thus, $\sqrt{16}$, or $\sqrt[2]{16}$ means the square root of 16, $= 4$, and $\sqrt[3]{8}$ means the cube root of 8, $= 2$.

EXERCISE VII.

1. Find the 4th root of 625.

Since $5 \times 5 \times 5 \times 5 = 625$, $\sqrt[4]{625} = 5$.

Find the value of:

- | | | |
|--------------------|--------------------|-----------------------------------|
| 2. $\sqrt{36}$ | 5. $\sqrt[5]{-32}$ | 8. $\sqrt[2]{(60 - 12 + 1)}$ |
| 3. $\sqrt[4]{81}$ | 6. $\sqrt[3]{-64}$ | 9. $\sqrt[3]{(-80 + 19 - 64)}$ |
| 4. $\sqrt[3]{125}$ | 7. $\sqrt[4]{256}$ | 10. $\sqrt[4]{(1471 - 204 + 29)}$ |

COMPOUND NUMBERS.

64. A **Simple Number** is one which involves no indicated operations; as 6, -17 , 325.

65. A **Compound Number** is one which involves indicated operations; as $8 - 3$, $7 - 3 + 4$, $8 - 5 \times 4$.

66. To **Simplify** a compound number is to perform the indicated operations, and thus reduce it to a simple number.

67. The **Terms** of a compound number are its complementary parts.

Hence, to simplify a compound number, we must first simplify its terms. Thus, the terms of

$8 + 6 \times 3 - 18 \div 2 - (13 - 4 + 2) + \sqrt{49}$,
are $+ 8$, $+ 6 \times 3$, $- 18 \div 2$, $-(13 - 4 + 2)$, and $\sqrt{49}$. Simplifying these, we obtain 8, $+ 18$, $- 9$, $- 11$ and 7, respectively. Now adding these, we obtain $+ 13$.

68. In general, the compound number within a parenthesis is a term, or the factor of a term; and hence, if any, should be simplified first.

Next, perform the operations indicated by $\times$ and $\div$, as products and quotients are generally terms.

Then, perform the operations denoted by $+$ and $-$.

EXERCISE VIII.

Simplify:

1. $4 \times 6 + 3$.
2. $5 - 3 \times 7$.
3. $2 \times 5 - 4$.
4. $3 + 8 \times 6$.
5. $8 \div 2 - 4$.
6. $6 + 9 \div 3$.
7. $12 - 6 \times 3 \times 5$.
8. $15 \div 3 + 7 + 4$.
9. $25 - 4 \times 3 + 6 \div 2$.
10. $25 \times (12 - 4 + 2)$.
11. $(16 + 10 - 2) \div 4 + 3$.
12. $(16 + 4) \div \sqrt{(30 - 5)}$.
13. $(6 \div 2 + 7) \times [8 \div (5 \times 3 - 11) + 3 - 6 \div 2]$.
14. $[12 - 15 \div 5 - (9 - 2 \times 4)] \div \frac{1}{2}$ of $(12 - \sqrt{5} \times 4 - 4)$.

The signs $\times$ and $\div$ indicate opposite operations just as $+$ and $-$ do. Hence, when several numbers are connected by the signs $\times$ and $\div$, as $24 \div 6 \times 3$, we may simplify the expression by dividing the product of those preceded by $\times$ by the product of those preceded by $\div$. In such cases, if there is no sign before the first factor, $\times$ is understood.

Simplify:

15. $6 \div 3 \times 2$.
16. $6 \times 3 \div 2 \div 4$.
17. $\div 8 \times 12 \times 4 \div 6$.
18. $24 \div 3 \times 5 \div 10 \times 7 \div 14$.

CHAPTER II.

GENERAL OR ARBITRARY NUMBERS.

ELEMENTARY PRINCIPLES AND DEFINITIONS.

69. The second particular in which Algebra is an extension of Arithmetic (**11**) is this:

Arithmetic treats of particular or special numbers denoted by figures, and Algebra treats of these and also of general or arbitrary numbers denoted by letters; as a , b , c , etc.

70. A **General or Arbitrary Number** is one which may have any reasonable value.

ILLUSTRATIONS.

1. If we say a boy is n years old, n is an arbitrary number, and may have any reasonable value like 2, 3, 6, etc. The number 70 would not be a reasonable value, since it would be inconsistent to say a boy 70 years old.

2. From a rope 100 ft. long a piece b ft. long was cut. Here, b is an arbitrary number, and may have any value between 0 and 100, such as 5, 24, 70, etc. The number 110 would not be a reasonable value, as there were only 100 ft. in all.

3. James has \$ a , John \$ b and both have \$7. Here, the general numbers a and b may each have any value, algebraically, provided that their sum shall be 7; as, $a = 3$, $b = 4$; $a = 1$, $b = 6$; $a = 11$, $b = -4$; $a = -13$, $b = 20$; etc.

71. General numbers are multiplied by writing the factors side by side with no sign between them.

Thus, 5 times a is written $5a$; m times n , mn ; and b times $(x + y)$, $b(x + y)$.

72. Multiplication is sometimes indicated by a dot or point ($\cdot$)

Thus, $a \cdot b \cdot c$ means $a \times b \times c$, or abc .

73. A Coefficient is the known factor of a product which shows how many times the other factor is taken.

Thus, in $7a$, 7 is the coefficient of a ; in $7ay$, 7 is the coefficient of ay , or if a be known, $7a$ is the coefficient of y . When no numerical coefficient is written, 1 is understood. Thus, ay means $1ay$.

74. A coefficient is the result of addition, and an exponent is the result of multiplication (**59**).

Thus

$$5a = a + a + a + a + a;$$

$$a^5 = a \times a \times a \times a \times a.$$

$$\text{If } a = 2, 5a = 2 + 2 + 2 + 2 + 2 = 10;$$

$$\text{and } a^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32.$$

EXERCISE IX.

What is the meaning:

1. Of $4c$? Ans. $c + c + c + c$, or 4 times c .

2. Of $6a^3b^2c$? Ans. $6 \times a \times a \times a \times b \times b \times c$.

3. Of $7ab^3n$?

5. Of $3n^4gy^3$?

4. Of $5a^2c^4x^2$?

6. Of $5(a + b)x^2y$?

75. Factors expressed by numbers are called *numerical*, when expressed by letters, *literal* factors.

76. In Algebra, the **Symbols**:

1^o, Of **Numbers**, are positive and negative figures and letters; as $+ 5$, $- 7$, $+ a$, $- n$.

2^o, Of **Operations**, are $+$, $-$, $\times$, $\div$, $\sqrt{}$, exponents.

3^o, Of **Aggregation**, are the bar, $|$; the vinculum, — ; the parenthesis, $(\)$; the brackets, $[\]$; and the brace, $\{ \}$.

Thus, each of these expressions $a + b$ signifies that $a + b$ is to be treated as one number. $\left. \begin{array}{l} a \\ + b \end{array} \right|$; $\overline{a + b}$; $(a + b)$; $[a + b]$; $\{a + b\}$.

77. An **Algebraic Number** is any quantity expressed in algebraic symbols of numbers, operations and aggregations.

Thus the following are algebraic numbers:

1^o, $5a$, which is 5 times the number denoted by a .

2^o, $4ab + 3(b - c)^2$, which is 4 times the product of the numbers denoted by a and b , increased by 3 times the square of the difference between the numbers denoted by b and c .

78. When an algebraic number is made up of parts connected by the signs $+$ and $-$, each of these parts is called a **Term** (67).

Thus, in the number $a + 2bc - 4n^3$, there are three terms, a , $2bc$ and $- 4n^3$.

79. A **Monomial** is an algebraic number of one term; as $4a^2b$, $- a^2b$, $+ 7a^3c^2x$.

80. A **Binomial** is an algebraic number of two terms; $a + b$, $5x - 3c^2$.

81. A Trinomial is an algebraic number of three terms; as $a - b + c$, $a^2 - 7bc^2 + mn^3x$.

82. A Polynomial is an algebraic number of two or more terms; as $a^2 - 5ax + 7c^3 + bd$.

A polynomial may be regarded as a compound number (65).

83. Like or Similar Terms are those containing the same powers of the same literal factors; as a^3c , and $7a^3c$, $5a^2(b - c)^2$ and $8a^2(b - c)^2$.

84. The Degree of a term is the number of its literal factors, or the sum of the exponents of its literal factors. Thus, $3ab$ is of the second degree, $5a^2b$, $7abc$, and $8c^3$ are of the third degree.

85. A Homogeneous Polynomial has all its terms of the same degree; as $5x^2 - 3xy$; $4a^3c - 3x^4 + 7x^2y^2$.

86. A Polynomial is arranged according to the powers of some letter, when the exponents of that letter either descend or ascend in order of magnitude.

Thus, $ax^3 + bx^2 - 5cx + d$ is arranged according to the descending powers of x ; and $5y - 6y^2 + 7y^3$, according to the ascending powers of y .

87. Two numbers are Reciprocals of each other when their product is unity; as $\frac{2}{3}$ and $\frac{3}{2}$, a and $\frac{1}{a}$.
The reciprocal of a number is that number inverted.

NUMERICAL VALUE.

88. The **Numerical Value** of an algebraic number is the number obtained by giving a particular value to each letter, and then performing the operations indicated (68).

EXERCISE X.

If $a = 5, b = 4, c = 3, d = 2, e = 1, x = 0$, find the numerical values of the following algebraic numbers:

1. $7a + 6b - 3d$.
2. $8b - 6x + 7c - 5a$.
3. $ab + ce - ad + cx$.
4. $a^2b - b^2c + d^2x$.
5. $a^3 + d^3 - b^2 + c^2$.
6. $a^2 - (b^2 + c^2 - ad)$.
7. $(abc + bcd) \div cde$.
8. $10c^4 - 6a^2x + 5e^3$.
9. $\frac{a^3 - b^3}{a - b} + \frac{c^2 - d^2}{c + d}$.
10. $\frac{a^4 - d^4}{a^2 + d^2} + \frac{b^3 + e^3}{b + e} - \frac{a^3 - 8}{a - 2}$.
11. $\frac{(a - d)^2 - (a - b + c)^2}{(a - d) - (a + b - c)} + \sqrt{(a - c) \times d}$.
12. $(a^e - c^e) \times (b^d - c^d) \div (d^a - e^a)$.

89. In Algebra, the **Symbols** :

1^o, Of **Relation**, are $=$, $>$ and $<$, which stand for "is equal to," "is greater than," and "is less than," respectively.

Thus, $5 + 3 = 8$; $4 \times 5 > 18$; $12 \div 3 < 5$.

2^o, Of **Deduction**, are $\therefore$ and $\because$, which stand for "hence" and "because," respectively.

Thus, $2 + 3 = 5$, $\therefore 3 = 5 - 2$. $6 \div 3 = 2$, $\because 2 + 3 = 5$.

30. Of **Continuation**, are dots, ..., or dashes, — —, which are read, “and so on.”

Thus, $2 + 3 + 4 \dots 7$; means $2 + 3 + 4 + 5 + 6 + 7$, or 27.

90. The product $1 \times 2 \times 3 \dots 10$, is indicated by writing the last factor 10.

That is, $\underline{6} = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$.

STATEMENTS.

91. A **Statement** is the expression of some relation between numbers.

92. A statement is expressed in **Common Language**, when the relation and numbers are written in *words*; and in **Algebraic Language**, when the relation and numbers are expressed in *algebraic symbols*.

Thus, “eight and seven are fifteen” is a statement expressed in common language, and “ $8 + 7 = 15$ ” is an expression of the same statement in algebraic language.

93. CASE I. — *To translate a statement from common to algebraic language.*

EXERCISE XI.

1. The sum of two positive numbers is greater than their difference.

Solution: To translate this statement, we first assume letters to denote the numbers. Let a and b be the letters, then the statement becomes

$$(a + b) > (a - b).$$

94. Rule.—*For the words, substitute the letters and signs which denote the numbers, the relations and the operations to be performed.*

2. The sum of any two numbers increased by their difference is equal to two times the greater number.
3. One-half of the sum of any two numbers, diminished by one-half of their difference, is equal to the smaller of the two numbers.
4. The sum of any two numbers multiplied by their difference, is equal to the difference of their squares.
5. The square of the sum of two numbers, is equal to the sum of their squares increased by two times their product.
6. The square of the difference of two numbers, is equal to the sum of their squares diminished by two times their product.

95. CASE II.—*To translate a statement from algebraic to common language.*

EXERCISE XII.

1. $\frac{1}{2}(a + b) + \frac{1}{2}(a - b) = a.$

Solution: One-half of the sum of any two numbers, increased by one-half of their difference, is equal to the larger of the two numbers.

96. Rule.—*For the letters representing numbers, and the signs indicating relations and operations, substitute words.*

2. $(a + 1)(a - 1) = a^2 - 1.$

3. $(a + 3)(a - 3) = a^2 - 9.$

4. $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2).$

5. $(a + b)^2 - (a - b)^2 = 4ab.$

6. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
 7. $4(a + b) = 4a + 4b$.
 8. $\frac{a - b}{3} = \frac{a}{3} - \frac{b}{3}$.
 9. $\sqrt{a \times b \times c} = \sqrt{a} \times \sqrt{b} \times \sqrt{c}$.
 10. $(a \times b \times c)^3 = a^3 \times b^3 \times c^3$.

ADDITION.

97. Principles.—1°, *The sum of two or more numbers is the same in whatever order they are added.*

1°, *Similar terms only may be united into one.*

EXERCISE XII.

1. Add $5a^2 + 3ab - 6d$; $-a^2 - 5ab + 2d$;
 $2a^2 + 3d - 5x^2y$; $-2ab - 3x^2y$.

Solution. For convenience, we write similar terms in the same column, and then add the coefficients (25.)

$5a^2 + 3ab - 6d$
$- a^2 - 5ab + 2d$
$2a^2 \qquad + 3d - 5x^2y$
$\qquad - 2ab \qquad - 3x^2y$
$6a^2 - 4ab - d - 8x^2y$

98. Rule. I. *Unite similar terms by prefixing the sum of the coefficients to the common letters.*

II. *Connect dissimilar terms by their signs.*

2. Add $4a - 3b$; $6a + 9b$; $- 2a - 6c$; $- 8b + 10c$.
 3. Add $7a^2 - 6bc$; $8a^2 + 7bc$; $+ 6bc - 5$; $+ a^2 - 7$.
 4. Add $8a^3 - 6a^2$; $- 8a^2 + 5a$; $11a + 6$; $- 3a^3 - 5$.
 5. Add $5a^4 + 3a^3 - 8a^2$; $- 3a^3 + 5a^2 - 11a$; $+ 3a^2 - 9a + 7$.
 6. Add $4xy^2 + 3x^2y$; $5x^2y - 2xy^2$; $7x^2y^2 - 6x^2y$.

7. Add $9a^3b, -8a^2b^2, +6a^3b, -7ab^3, +9a^2b^2,$
 $-5a^3b, +4ab^3.$
8. Add $a^3 + 3a^2b, -3a^2b - b^3, 2b^3 + 4a^2b,$
 $-a^3 - 2a^2b.$
9. Add $3x^2y - 4xy^2 + 3x^2, +2xy^2 - 5x^2 + 7x^2y,$
 $-5x - 8x^2 + 7xy^2 - 4y + 6x^2y - 5x + 11x^2$
 $+ 8y, 10x^2 - 2x - 3x^2y^2, -7x^2y + 2x + 3.$
10. Add $5\sqrt{ab}$ and $7\sqrt{ab}.$ Ans. $12\sqrt{ab}.$
11. Add $6\sqrt{abc}, -4\sqrt{abc}, +8\sqrt{abc}.$
12. Add $5(a - b), 8(a - b), 3(a - b).$

Add:

13. $6\sqrt{ac} + 7\sqrt{m} - 5x; 4\sqrt{ac} - 4\sqrt{m} + 7x.$
14. $7\sqrt{a - b} - 8\sqrt{a + c}; -5\sqrt{a - b}$
 $- 6\sqrt{a + c}.$
15. $3a^2 - 6(a - n)^2; -9a^2 + 11(a - n)^2.$
16. $5xy + 4x^2z - 3xz^2, 7xy + 9x^2z - xz^2,$ and
 $-10xy + 20x^2z - 15xz^2.$
17. $3a + b - c, 9a - 6b - 10c + f, 8a + 6b$
 $- 8c + 3f, 4a - b + c + g,$ and $6a - 3b - 8c.$
18. $(a - c)^2 + 8a\sqrt{6} - 4c^3x; 5(a - c)^2 - 10a\sqrt{6}$
 $- 3c^3x.$
19. Find the sum of $8a^2 - 3ab + 5ac - 3(a + b)$
 $+ 4\sqrt{2}, -6ac + 7ab - 3\sqrt{2},$ and $-4ab$
 $- 6a^2 + 5(a + b) - \sqrt{2}.$

SUBTRACTION.

99. To **Reverse** a number is to change its sign.

100. Principle.—*Subtracting a number is equivalent to adding the reverse of that number (37).*

101. Rule.—*Change the signs of the numbers to be subtracted, or conceive them to be changed, and unite the terms as in addition.*

EXERCISE XIV.

1. From $+ 5a^2 - 6ab - 7b^2$ take $+ 3a^2 + 5ab - 9b^2 - 5$.

Solution.—For convenience we write the subtrahend under the minuend, placing similar terms in the same column, and proceed according to the rule.

$$\begin{array}{r} + 5a^2 - 6ab - 7b^2. \\ + 3a^2 + 5ab - 9b^2 - 5. \\ \hline + 2a^2 - 11ab + 2b^2 + 5. \end{array}$$

2. From $x^2 - 5x - 6$ take $3x^2 - 5x + 3$.

3. From $3b + 7a - 6c$ take $3c - 4b + 5a$.

4. From $5an - 7ax + 8$ take $- 6 - 7an + 5ax$.

From:

5. $6ab$ take $4ab$,

9. m take n .

6. $9x^2y$ take $- 3x^2y$.

10. m take $- n$.

7. $- 7ab^2$ take $5ab^2$.

11. $- m$ take n .

8. $- 8xz$ take $- 3xz$

12. $- m$ take $- n$.

How much more:

13. Is $5a^2x$ than $- 6a^2x$?

14. Is $- 9xy$ than $5xy$?

15. Is $7\sqrt{3}$ than $4\sqrt{3}$?

16. Is $9\sqrt{a}$ than $- 3\sqrt{a}$?

17. Is $a + b$ than $a - b$?

18. Is $5a - 3x + 2y$ than $- 5a - 5y + 7x$?

19. From $7c$ take $-3c$?

20. From $7\sqrt{m}$ take $-3\sqrt{m}$?

What number must be added:

21. To $5x^2$ to make $8x^2$?

22. To a to make b ?

23. To $-c$ to make $-a$?

24. To $-5a$ to make $3a$?

25. To $a^2 + b$ to make $a^2 - b$?

26. To $x - y$ to make $x + y$?

27. To $x^2 - 6x + 17$ to make $-5x^2 + 7x + 15$?

28. To $3\sqrt{a} - 6\sqrt{b} + 2\sqrt{5}$ to make $5\sqrt{a} - 8\sqrt{b} - 3\sqrt{5}$?

Reduce to simplest terms (38):

29. $2a + 3b - (4a - 5b + c)$.

30. $3a^2 - 2bx + x^2 - (5bx + 4a^2 - 2x^2)$.

31. $(a + b) + (b + c) - (a + c)$.

32. $(3a - 2b - c) - (a - 2b + 3c)$.

33. $(2x^2 + y) - (2y - 3b) - (5b - 3x^2)$.

34. $1 - (1 - x) + (1 - x + x^2) - (1 - x + x^2 - x^3)$.

35. $3a - [2b - (5a - 3b) - a]$.

36. $5x - [4x - (3x - 2x) - x]$.

37. $7a - [3a - \{4a - (5a - 2a)\}]$.

38. $2x - [7x - (6x + 1)] + [8x - (6x - 1)]$.

39. $3\sqrt{2} - [5\sqrt{2} - (4\sqrt{2} - 2\sqrt{2}) + 7\sqrt{2}]$.

102. If we wish to enclose any number of the terms of a polynomial within a parenthesis, we may place:

1°. The sign $+$ before the whole;

2°. The sign $-$ before the whole, provided the sign of every term within the parenthesis be changed.

Thus, $a - b + c - d = a - (b - c + d)$, or $(a - b) + (c - d)$.

Change the polynomial

$$a - b + 2c - 3d + 4e - 5f + x - 5mn,$$

into a binomial by enclosing within parentheses:

40. The first two terms, and the last six terms.
41. The first three terms, and the last five terms.
42. The first four terms, and the last four terms.
43. The first five terms, and the last three terms.
44. The positive terms, and the negative terms.
45. From the sum of a and b take their difference.
46. From the difference of a and b take their sum.
47. From the sum of $x^2 - 3x + 5$ and $2x^2 + 7x - 4$, take their difference.
48. From the difference of $a^2 + 3ab - 5b^2$ and $b^2 + 3ab - 5b^2$, take their sum.

MULTIPLICATION.

103. Principles.—1°. *The product of several factors is the same in whatever order they are multiplied.*

2°. *Like signs produce +, unlike signs —.*

104. Multiplication of monomials.

EXERCISE XV.

1. Multiply $+ 6a^3b^2y$ by $- 4a^2b$.

Solution. $+ 6a^3b^2y \times - 4a^2b = + 6aaaabbb y \times - 4aab$
 $= + 6 \times - 4 aaaaaabbb y = - 24a^5b^3y =$
 $- 24a^3 + 2b^2 + 1y.$

105. Rule.—*To the product of the numerical coefficients annex the literal factors, giving each an exponent equal to the sum of all its exponents in the several factors.*

Multiply:

- | | |
|--------------------------------|---------------------------------|
| 2. $6a^4$ by $3a^2$. | 8. a^2 by a^3 . |
| 3. $7a^3$ by $5a^2b$. | 9. a^n by a^m . |
| 4. $8a^2b$ by $-3a^5x$. | 10. a^{3x} by a^{2x} . |
| 5. $-9ab^3$ by $-5b^2x^2$. | 11. a^{c+n} by a^{c-n} . |
| 6. $8c^2n^2x$ by $7a^3x^4$. | 12. $-7a^2$ by $3a^n$. |
| 7. $+9x^2y^3z$ by $-4x^3z^5$. | 13. $-5a^2c^n$ by $-3a^m c^5$. |

Find the value of:

14. $+9a^3b^4c^2x \times 8a^5bc^4n^2x$.
15. $-7a^nb^mc \times +6a^{2n}b^{3m}c^x$.
16. $-6(a+b)^2x^3 \times -10(a+b)^3x^4$.
17. $+11(a+b)^3(a-b)^4 \times -7(a+b)^2(a-b)^5$.
18. $-16(a+2)^3(a+1)^2(a+b) \times -2(a+2)^4(a+1)^3(a-b)^2$.
19. $+7(a-b+c)^3 \times -5(a-b+c)^2$.
20. $+3a^3b \times -2a^2b^2c \times -3c^3x^2 \times 3b^5y$.

106. Multiplication of polynomials by monomials.

21. Multiply $(8+5)$ by 3.

$$\begin{aligned}
 (8+5) \times 3 &= (8+5) + (8+5) + (8+5) \\
 &= 8+8+8+5+5+5 \\
 &= 3 \times 8 + 3 \times 5 = 39.
 \end{aligned}$$

107. Principle.—*A number is multiplied by multiplying its complementary parts.*

108. Hence, Rule.—*Multiply each term of the polynomial by the monomial, and add the partial products.*

22. Multiply $3a^2b - 5ab^2c - 2ax$ by $4ab^2$.

Solution: $3a^2b - 5ab^2c - 2ax$.

$4ab^2$.

$12a^3b^3 - 20a^2b^4c - 8a^2b^2x$.

Multiply:

23. $5a^3 - 3a^2 + 4a - 2$ by $4a^2$.

24. $7x^4 + 5x^3 - 3x^2 - 8x$ by $-5x^3$.

25. $a^3x - 3a^2x^2 + 3ax - 1$ by $4a^2x$.

26. $3a^3 - 6a^2b + 7ab^2 - 2b^3$ by $-3a^2b^2$.

27. $c^5 - bc^4 + b^2c^3 - b^3c^2$ by a^2b^3c .

28. $-7a^5 + 6a^4x - 5a^3x^2 + 4a^2x^3$ by $-2a^2x^3y$.

29. $5(a+b)^3 + 6(a+b)^2$ by $3(a+b)^2$.

30. $6a^2(a-b)^3 - 7a^3(a-b)^4$ by $-5a^4(a-b)^3$.

31. $7(x+y)^3(x-y)^2 - 9(x+y)(x-y)^5$ by $4(x+y)^4(x-y)^5$.

32. $5a^n - 6a^m + 7a^5$ by $4a^2$.

33. $-5a^3 + 12a^7 - 3a^n$ by $-3a^n$.

34. $-6a^3b^2 + 10a^5b^n - 7a^nb^5 + 3a^{2n}b^{3n}$ by $9a^nb^{2n}$.

109. Multiplication of polynomials by polynomials.

35. Multiply $(8 + 5)$ by $(7 - 3)$.

$$\begin{aligned} (8 + 5) \times (7 - 3) &= 13 \times (7 - 3), \text{ (Art. 107.)} \\ &= 7 \times 13 - 3 \times 13 = 7 \times (8 + 5) - 3 \times (8 + 5) \\ &= 7 \times 8 + 7 \times 5 - 3 \times 8 - 3 \times 5. \end{aligned}$$

110. Principle.—*A number is multiplied by multiplying each of its comp. parts by each of the comp. parts of the multiplier.*

111. Hence, Rule.—*Multiply each term of one factor by each term of the other, and add the partial products.*

36. Multiply $6 - 4 + 5$ by $8 - 3$.

1° Solution.

$$6 - 4 + 5 = 7;$$

$$8 - 3 = 5;$$

$$7 \times 5 = 35, \text{ Ans.}$$

2° Solution.

$$6 - 4 + 5$$

$$8 - 3$$

$$48 - 32 + 40$$

$$- 18 + 12 - 15$$

$$48 - 50 + 52 - 15 = 35, \text{ Ans.}$$

Let the next five examples be solved and verified in the same manner.

Multiply:

37. $9 + 7 - 5$ by $7 - 2$.

38. $5 - 6 + 3 - 7$ by $3 - 10$.

39. $10 - 4 + 3$ by $6 - 5 - 4$.

40. $2 - 4 + 3 - 7$ by $5 - 2 + 3$.

41. $2^2 + 3^2 - 2^2$ by $3^2 - 2^1$.

42. Multiply $-3x + 6 - 2x^2$ by $2x^2 - 3x$.

Solution—It is best always to

arrange both factors according to the ascending or descending

powers of some one of the letters. In this instance they

are arranged according to the ascending power of x .

$$+ 6 - 3x - 2x^2$$

$$- 3x + 2x^2$$

$$- 18x + 9x^2 + 6x^3$$

$$+ 12x^2 - 6x^3 - 4x^4$$

$$- 18x + 21x^2 - 4x^4, \text{ Ans.}$$

Multiply:

43. $a + b$ by $a + b$.

44. $a - b$ by $a - b$.

45. $a + b$ by $a - b$.

46. $5a + 3b$ by $5a + 3b$.

47. $7c - 6x$ by $7c - 6x$.

48. $8a^2 + 7b$ by $8a^2 - 7b$.

49. $a^2 + 2ab + b^2$ by $a^2 + 2ab + b^2$.

50. $x^2 - 4xy + 4y^2$ by $x^2 - 4xy + 4y^2$.

Multiply:

51. $a^2 + 6ac + 9c^2$ by $a^2 - 6ac + 9c^2$.
52. $x^2 + xy + y^2$ by $x - y$.
53. $a^2 - 3ay + 9y^2$ by $a + 3y$.
54. $a^8 + a^6 + a^4 + a^2 + 1$ by $a^2 - 1$.
55. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.
56. $a^2 + ab + b^2$ by $a^2 + ac + c^2$.
57. $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$ by $x - y - z$.
58. $a^2 + 2ab + b^2 + a + b + 1$ by $a + b - 1$.
59. $4x^4 - 12x^2y + 9y^2 - 2x^2 + 3y + 1$ by $2x^2 - 3y + 1$.

Find the products of:

60. $x - 3$, $x - 2$, $x + 1$, and $x + 3$.
61. $x - 4$, $x - 2$, $x + 2$, and $x + 4$,
62. $x + 2$, $x + 3$, and $x^2 - 6x + 6$.
63. $x - 4$, $x + 4$, and $x^2 + 16$.
64. $x^2 - x + 1$, $x^2 + x + 1$, and $x^4 - x^2 + 1$.
65. $x + a$, $x + b$, and $x + c$.
66. $5x - 6x^2 + x^3 - 24$ and $x^2 + 7 - 4x$.
67. $a^3 + 3ab^2 - b^3 - 3a^2b$ and $b^3 + a^3 + 3ab^2 + 3a^2b$.

Simplify:

68. $(a + b)^2$, or $(a + b)(a + b)$.
69. $(a + b)^2$, or $(a + b)(a + b)(a + b)$.
70. $(a + b)^4$, or $(a + b)(a + b)(a + b)(a + b)$.
71. $(x^2 + 2x)^2$.
72. $(x^3 - 3x^2)^2$.
73. $(2x + 3)^4$.
74. $(5a + 6c)^3$.
75. $(a^2 - 2)^4$.
76. $(2a^3 - 5a^2b)^2$.

Show that each of the following is equal to 0:

77. $(a + b)^2 - (a - b)^2 - 4ab$.
78. $(x + 1)^3 - (x - 1)^3 - 3(x + 1)^2 - 3(x - 1)^2 + 4$.
79. $(a + 1)^4 - 4(a + 1)^3 + 4(a + 1)^2 - 4(a - 1)^2 - 4(a - 1)^3 - (a - 1)^4$.

MULTIPLICATION OF BINOMIALS WHOSE FIRST TERMS ARE EQUAL.

112. I.—*When the second terms are positive.*

EXERCISE XVI.

1. Multiply $x + a$ by $x + b$.

Ans. $x^2 + (a + b)x + ab$.

113. Hence, Rule. 1^o, *Square the first term*; 2^o, *multiply the sum of the second terms by the first*; 3^o, *multiply the second terms*; 4^o, *add the three products.*

2. Multiply $x + 5$ by $x + 6$.

1^o, $x \times x = x^2$; 2^o, $(5 + 6) \times x = + 11x$; 3^o, $5 \times 6 = + 30$; 4^o, $x^2 + 11x + 30$, Ans.

Multiply:

3. $x + 3$ by $x + 7$.

8. $y + 7$ by $y + 20$.

4. $x + 6$ by $x + 8$.

9. $x^2 + 11$ by $x^2 + 5$.

5. $x + 7$ by $x + 13$.

10. $x^2 + 3a$ by $x^2 + 5a$.

6. $x + 12$ by $x + 5$.

11. $x^3 + 6b$ by $x^3 + 7b$.

7. $x + 9$ by $x + 10$.

12. $x^4 + 3n$ by $x^4 + 15n$.

114. II.—*When the second terms are negative.*

13. Multiply $(x - 5)$ by $(x - 9)$

1^o, $x \times x = x^2$; 2^o, $(-5 - 9) \times x = - 14x$; 3^o, $-5 \times -9 = + 45$; 4^o, $x^2 - 14x + 45$.

14. $x - 8$ by $x - 10$.

19. $y - 3$ by $y - 11$.

15. $x - 12$ by $x - 20$.

20. $x^2 - 11$ by $x^2 - 9$.

16. $x - 13$ by $x - 7$.

21. $x^2 - 5c$ by $x^2 - 8c$.

17. $x - 30$ by $x - 20$.

22. $x^3 - 10n$ by $x^3 - 7n$.

18. $x - 14$ by $x - 7$.

23. $x^4 - 5a^2c$ by $x^4 - 14a^2c$.

115. III.—*When the second terms have unlike signs, the positive term being the greater.*

24. Multiply $x + 8$ by $x - 3$.

1° , $x \times x = x^2$; 2° , $(+8 - 3)x = +5x$; 3° , $+8 \times -3 = -24$; 4° , $x^2 + 5x - 24$, Ans.

25. $x + 9$ by $x - 2$.

29. $x - 11$ by $x + 20$.

26. $x + 20$ by $x - 5$.

30. $x^2 - 8a$ by $x^2 + 12a$.

27. $x + 15$ by $x - 6$.

31. $x^2 + 10a$ by $x^2 - 7a$.

28. $x - 7$ by $x + 11$.

32. $x^3 + 12b$ by $x^3 - 10b$.

116. IV.—*When the second terms have unlike signs, the positive term being the less.*

33. Multiply $x - 11$ by $x + 4$.

1° , $x \times x = x^2$; 2° , $(-11 + 4)x = -7x$; 3° , $-11 \times 4 = -44$; 4° , $x^2 - 7x - 44$, Ans.

34. $x - 9$ by $x + 2$.

38. $x + 11$ by $x - 20$.

35. $x - 20$ by $x + 5$.

39. $x^2 + 8a$ by $x^2 - 12a$.

36. $x - 15$ by $x + 6$.

40. $x^2 - 10a$ by $x^2 + 7a$.

37. $x + 7$ by $x - 12$.

41. $x^3 - 12b$ by $x^3 + 10b$.

THREE SPECIAL CASES.

117. I.—*When the second terms are equal and positive.*

42. Multiply $x + a$ by $x + a$.

As before, we find $x^2 + 2ax + a^2$, Ans.; hence,
 $(x + a)^2 = x^2 + 2ax + a^2$. That is,

118. *The square of the sum of two numbers is equal to the sum of their squares + twice their product.*

119. II.—*When the second terms are equal and negative.*

43. Multiply $x - a$ by $x - a$.

As before we find $x^2 - 2ax + a^2$, hence,

$$(x - a)^2 = x^2 - 2ax + a^2. \text{ That is,}$$

120. *The square of the difference of two numbers is equal to the sum of their squares — twice their product.*

121. III.—*When the second terms are equal and have unlike signs.*

44. Multiply $x + a$ by $x - a$.

As before, we find $x^2 - a^2$, Ans.; hence,

$$(x + a)(x - a) = x^2 - a^2. \text{ That is,}$$

122. *The product of the sum and difference of two numbers is equal to the difference of their squares.*

123. A Formula is a general truth expressed by symbols.

The following examples should be solved by the preceding formulas and rules.

Simplify:

45. $(5 + 3)^2$.

53. $(a + 9)^2$.

46. $(5 - 3)^2$.

54. $(a - 8)^2$.

47. $(5 + 3)(5 - 3)$.

55. $(a + 9)(a - 9)$.

48. $(x + 6)^2$.

56. $(a^2 + 7)^2$.

49. $(x - 6)^2$.

57. $(a^2 - 5)^2$.

50. $(x + 6)(x - 6)$.

58. $(a^2 + 11)(a^2 - 11)$.

51. $(2x + 1)^2$.

59. $(3y - 1)^2$.

52. $(y + 4a)^2$.

60. $(z - 3a)^2$.

Simplify:

61. $(y^2 + 6a)^2$. 68. $(z^2 - 5a)^2$.
 62. $(x^2 + 7b)^2$. 69. $(x^2 - 9b)^2$.
 63. $(3x^2 + 2x)^2$. 70. $(4x^2 - 3x)^2$.
 64. $(5x^2 + 4ax)^2$. 71. $(5x^2 - 6bx)^2$.
 65. $(3ax + 6a^2)^2$. 72. $(3nx - 7n^2)^2$.
 66. $(5a^3b + 7ab^2)^2$. 73. $(6n^3x - 10n^2x^2)^2$.
 67. $(9 + x)(9 - x)$ 74. $(8ab + 2b^2)(8ab - 2b^2)$.
 75. $(5x^2 - 3y^2)(5x^2 + 3y^2)$.
 76. $(7a^3 + 4ab)(7a^3 - 4ab)$.
 77. $(a^2b^3 - ab^4)(a^2b^3 + ab^4)$.
 78. $(1 + 12a^4)(1 - 12a^4)$.
 79. $(5a^3x^2y - 1)(5a^3x^2y + 1)$.
 80. $(a^2 + 3)(a^2 - 3)(a^4 + 9)$.

SQUARING OF POLYNOMIALS.

EXERCISE XVII.

124. 1. Multiply
- $a + b + c$
- by
- $a + b + c$
- .

$$\begin{array}{r}
 a + b + c \\
 a + b + c \\
 \hline
 a^2 + ab + ac \\
 \quad ab \quad + b^2 + bc \\
 \quad \quad ac \quad + bc + c^2 \\
 \hline
 a^2 + 2ab + 2ac + b^2 + 2bc + c^2 \\
 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.
 \end{array}$$

125. Hence, Rule.—1°, *Square each term*; 2°, *take twice the product of each term by each of the terms that follow it*; 3°, *add all these results*.

2. Simplify
- $(3x^2 + 2x - 4)^2$

$$\begin{array}{l}
 1^\circ, \quad 3x^2 \times 3x^2 = +9x^4, \quad +2x \times 2x = +4x^2, \quad (-4) \times (-4) = +16; \\
 2^\circ, \quad 2(3x^2)(2x) = 12x^3, \quad 2(3x^2)(-4) = -24x^2, \quad 2(2x)(-4) = -16x; \\
 3^\circ, \quad 9x^4 + 4x^2 + 16 + 12x^3 - 24x^2 - 16x = 9x^4 + 12x^3 - 20x^2 - 16x + 16. \text{ Ans.}
 \end{array}$$

Simplify:

- | | |
|------------------------------|----------------------------------|
| 3. $(6 - 4 + 2)^2$. | 10. $(a - b + c)^2$. |
| 4. $(9 - 5 + 6)^2$. | 11. $(a - b - c)^2$. |
| 5. $(7 - 5 + 3 - 1)^2$. | 12. $(a^2 + a - 4x)^2$. |
| 6. $(x^2 - 2x)^2$. | 13. $(2x^3 - 3x^2 + 4x - 1)^2$. |
| 7. $(x^2 + 2x - 3)^2$. | 14. $(2 - a + 2a^2 - a^3)^2$. |
| 8. $(x^3 - 3x^2 - 5)^2$. | 15. $(a^2 - 2ab + 6b^2)^2$. |
| 9. $(2a^2 - 3ab + 4b^2)^2$. | 16. $(a^3 - 3a^2 + 3a - 1)^2$. |

DIVISION.

126. Since *Dividend* = *Divisor* $\times$ *Quotient*, we have the following principles:

1°, *Multiplying or dividing the dividend multiplies or divides the quotient.*

2°, *Multiplying or dividing the divisor divides or multiplies the quotient.*

3°, *Multiplying or dividing both dividend and divisor by the same number does not affect the quotient.*

127. Division of Monomials.

Divide $-6a^4b^3c$ by $+2a^2b$.

$$\begin{aligned} \frac{-6a^4b^3c}{+2a^2b} &= \frac{-6aaaabbbbc}{2aab} = -3aabbcc \\ &= -3a^2b^2c, \text{ which } = \frac{-6}{+2} a^{4-2} b^{3-1} c. \end{aligned}$$

Rule.—*Divide the coefficient of the dividend by that of the divisor, and annex to the quotient the literal factors, giving to each an exponent found by subtracting its exponent in the divisor from its exponent in the dividend.*

EXERCISE XVIII.

Divide:

1. $12a^5b^2$ by $4a^2b$.
2. $15c^2x^4$ by $-5c^2x^3$.
3. $10a^4y$ by $2ay$.
4. $18a^6c^3$ by $-6a^2c^2$.
5. $45a^3n^5$ by $9an^2$.
6. $-22b^5d^4$ by $-11b^2d$.
7. $-24m^2n^3$ by $8m^2$.
8. $30a^3b^4c$ by $-5a^2b^3$.
9. $8a^5b^4c^3x$ by $4a^2b^3x$.
10. $-a^7b^6c^5x^4$ by $-a^6b^6x^3$.
11. Divide a^2 by a^2 .

1° a^2 contains a^2 1 time, $\therefore a^2 \div a^2 = 1$.

2° By the rule, $a^2 \div a^2 = a^{2-2} = a^0$. $\therefore a^0 = 1$.

In the same manner divide:

12. b^3 by b^3 .
13. c^5 by c^5 .
14. $(a+b)^4$ by $(a+b)^4$.
15. x^n by x^n .

128. Hence, *Any number raised to the power of 0 is equal to 1.*

129. *Division of Polynomials by Monomials.*

Since $(a - b + c)n = an - bn + cn$,

$$(an - bn + cn) \div n = a - b + c.$$

130. Principle.—*A number is divided by dividing each of its comp. parts.*

131. Hence, Rule.—*Divide each term of the polynomial by the monomial.*

16. Divide $9x^3y^4 - 12x^2y^5 + 30x^4y^3$ by $3x^2y^3$.

$$\text{Solution: } 3x^2y^3 \overline{) 9x^3y^4 - 12x^2y^5 + 30x^4y^3} \\ 3xy - 4y^2 + 10x^2, \text{ Ans.}$$

Divide:

17. $16 - 24 + 40$ by 8 .
18. $36 + 45 - 63 - 9$ by -9 .
19. $16a^2 - 32a^3b + 40a^4c$ by $-8a$.
20. $20a^3x^2y + 60a^4x^3y^2 - 10a^2x^2$ by $10a^2x^2$.
21. $5a^4b^2x - 7a^5b^4xy + 3a^6b^5nx^2$ by $-a^4b^2x$.
22. $70a^3c^2m + 98a^2c - 140a^4c^2b$ by $14a^2c$.
23. $-16a^4b^3c^2 + 24a^2b^6c^3 - 32a^2b^2c^8$ by $-8a^2b^2c^2$.
24. $16(a+b)^5 - 8(a+b)^3 + 4m(a+b)^2$ by $4(a+b)^2$.
25. $12a^2(a-x)^2 + 9ac(a-x)^5 - 15a(a-x)$ by $-3a(a-x)$.
26. $36(a-x)^3(b+y)^2 - 6c(a-x)(b+y)^3$ by $-6(a-x)(b+y)$.

132. *Division of Polynomials by Polynomials.*

In order to illustrate the principles and successive steps, let us multiply one polynomial by another, and then divide the product, regarded as a dividend, by one of the factors, regarded as a divisor.

Multiply $x^2 + 2xy - y^2$ by $x^2 - 5xy + 3y^2$

(a)..... $x^2 + 2xy - y^2$

(b)..... $\underline{x^2 - 5xy + 3y^2}$

(c)..... $x^4 + 2x^3y - x^2y^2$

(d)..... $-5x^3y - 10x^2y^2 + 5xy^3$

(e)..... $\underline{+ 3x^2y^2 + 6xy^3 - 3y^4}$

(f)..... $x^4 - 3x^3y - 8x^2y^2 + 11xy^3 - 3y^4$

The three comp. parts of this result are the partial products (c), (d), (e); of which (c) = (a) $\times$ x^2 , (d) = (a) $\times$ $-5xy$, (e) = (a) $\times$ $3y^2$.

Now let us divide (f) by (a).

$$\begin{array}{r}
 (f) = x^4 - 3x^3y - 8x^2y^2 + 11xy^3 - 3y^4. \quad |x^2 + 2xy - y^2 = (a) \\
 (c) \dots x^4 + 2x^3y - \quad x^2y^2. \quad \quad \quad x^2 - 5xy + 3y^2 = (b) \\
 (d) + (e) - 5x^3y - 7x^2y^2 + 11xy^3 - 3y^4. \\
 (d) \dots - 5x^3y - 10x^2y^2 + 5xy^3. \\
 (e) \dots \quad \quad \quad 3x^2y^2 + 6xy^3 - 3y^4. \\
 (e) \dots \quad \quad \quad 3x^2y^2 + 6xy^3 - 3y^4.
 \end{array}$$

The first term of (b) is the first term of (f) divided by the first term of (a); the second term of (b) is the first term of (d) + (e) divided by the first term of (a), &c. Hence:

133. Rule I. *To get the first term of the quotient divide the first term of the dividend by the first term of the divisor.*

II. *Multiply all the terms of the divisor by the first term of the quotient, and subtract the product from the dividend.*

III. *If there be a remainder, consider it as a new dividend and proceed as before.*

134. The dividend, divisor and remainders should always be arranged according to the ascending or descending powers of some common letter.

135. Write the final remainder, if any, over the divisor in the form of a fraction, and add it to the quotient.

27. Divide

$$\begin{array}{r}
 x^2 - 7x + 12 \text{ by } x - 3 \\
 x^2 - 7x + 12 \quad | \quad x - 3 \\
 + x^2 - 3x \quad \quad x - 4 \\
 \hline
 - 4x + 12 \\
 - 4x + 12 \\
 \hline
 \hline
 \end{array}$$

28. Divide

$$\begin{array}{r}
 x^2 + 3x - 54 \text{ by } x + 9 \\
 x^2 + 3x - 54 \quad | \quad x + 9 \\
 + x^2 + 9x \quad \quad x - 6 \\
 \hline
 - 6x - 54 \\
 - 6x - 54 \\
 \hline
 \hline
 \end{array}$$

29. Divide $6a^4 + 2a - 10a^3 - 2$ by $-2a + 1 + 3a^2$.

$$\begin{array}{r}
 6a^4 - 10a^3 + 0a^2 + 2a - 2 \quad | \quad 3a^2 - 2a + 1 \\
 + \underline{6a^4 - 4a^3 + 2a^2} \quad 2a^2 - 2a - 2 \\
 - 6a^3 - 2a^2 + 2a \\
 - \underline{6a^3 + 4a^2 - 2a} \\
 - 6a^2 + 4a - 2 \\
 - \underline{6a^2 + 4a - 2} \\
 0
 \end{array}$$

30. Divide $1 + 7 + 12$ by $1 + 3$.

31. Divide $1 - 8 + 15$ by $1 - 5$.

32. Divide $1 + 9 + 26 + 24$ by $1 + 3$.

33. Divide $1 - 9 + 26 - 24$ by $1 - 6 + 8$.

34. Divide $5^3 - 2^3$ by $5 - 2$.

Divide:

35. $x^2 + 11x + 30$ by $x + 5$.

36. $x^2 - 11x + 28$ by $x - 4$.

37. $x^2 + 2x - 35$ by $x + 7$.

38. $x^2 - 5x - 36$ by $x - 9$.

39. $x^2 - 49$ by $x + 7$.

40. $x^2 - 12ax + 27a^2$ by $x - 3a$.

41. $x^2 + 9a^2bx - 22a^4b^2$ by $x + 11a^2b$.

42. $x^3 - 27$ by $x - 3$.

43. $x^4 - 16$ by $x^2 + 4$.

44. $a^2y^4 - 9b^2x^6$ by $ay^2 - 3bx^3$.

45. $a^3 - b^3$ by $a - b$.

46. $a^4 - b^4$ by $a - b$.

47. $x^5 - 2x^4 + 5x^3 + 3x^2 - 12x - 28$ by $x - 2$.

48. $a^4 - x^4 + 2ar^3 - a^2x^2$ by $ax + a^2 - x^2$.

49. $x^5 + x^4y - 5x^3y^2 + 6xy^4 + 2y^5$ by $x^2 + 3xy + y^2$.

Divide :

50. $2a^4 + 11a^3x + 20a^2x^2 + 13ax^3 + 2x^4$ by $a^2 + 3ax + 2x^2$.
51. $x^4 - y^4$ by $x + y$.
52. $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.
53. $100a^5 - 440a^4c + 235a^3c^2 - 30a^2c^3$ by $5a^3 - 2a^2c$.
54. $a^4 - b^4$ by $a^3 + a^2b + ab^2 + b^3$.
55. $6a^6 - 6y^6$ by $2a^2 - 2y^2$.
56. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$ by $a^3 - 3a^2b + 3ab^2 - b^3$.
57. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.
58. $16a^8b^4 + 4a^6b^6 + a^4b^8$ by $4a^4b^2 - 2a^3b^3 + a^2b^4$.
59. $16a^{12}b^8c^4 - a^4b^8c^{12}$ by $2a^8b^2c + ab^2c^3$.
60. $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.
61. $a^4 - 2a^2b^2 + b^4$ by $a^2 + 2ab + b^2$.
62. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.
63. $z^4 - 24z^3 + 206z^2 - 744z + 945$ by $z^2 - 8z + 15$.
64. $5a^7 - 22a^6b + 12a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5$ by $a^3 - 4a^2b + 2b^3$.
65. $x^4 + x^3 + x^2 + x + 1$ by $x^2 - 2x + 1$.

Divide (without simplifying terms):

66. $(x + a)^2 + 7(x + a) + 12$ by $(x + a) + 3$.
67. $(x - c)^2 - 9(x - c) + 20$ by $(x - c) - 4$.
68. $(x + n)^2 - 3(x + n) - 28$ by $(x + n) - 7$.
69. $(x + n)^2 - 25a^2$ by $(x + n) + 5a$.
70. $(x + a)^3 - 64$ by $(x + a) - 4$.
71. $(x + a - c)^2 + 5(x + a - c) - 24$ by $(x + a - c) - 3$.
72. $(x + c)^4 - (x + n)^4$ by $(x + c)^2 + (x + n)^2$.

· FINDING THE REMAINDER WITHOUT DIVIDING.

136. In certain cases the remainder may be found without performing the operation of division. The two following cases are the bases of so many important formulas and principles, that they are here presented under the forms of Propositions.

137. PROP. I.—*If a quantity containing y , can be divided by $y - a$ so as to leave a remainder which does not contain y , the remainder will be the same as the quantity itself with a written in the place of y .*

Thus, if we divide $y - a$ into $y^2 + Ay + B$,
the remainder will be..... $a^2 + Aa + B$.

PROOF. Let $y^3 + my^2 + ny + r$ represent the quantity to be divided by $y - a$. Denote the quotient by Q , and the remainder, which does not contain y , by R . Then,

$$y^3 + my^2 + ny + r = Q(y - a) + R.$$

Now, writing a in the place of y , we have

$$a^3 + ma^2 + na + r = Q \times 0 + R,$$

$$\text{or } a^3 + ma^2 + na + r = R.$$

138. PROP. II.—*If a quantity containing y , can be divided by $y + a$ so as to leave a remainder which does not contain y , the remainder will be the same as the quantity itself, with $-a$ written in the place of y .*

Thus, if we divide $y + a$ into $y^2 + Ay + B$,
the remainder will be..... $a^2 - Aa + B$.

Proof. Let $y^3 + my^2 + ny + r$ represent the quantity to be divided by $y + a$. Denote the quotient by Q , and the remainder, which does not contain y , by R .

Then

$$y^3 + my^2 + ny + r = Q(y + a) + R.$$

Writing $-a$ in the place of y , we have

$$(-a)^3 + m(-a)^2 + n(-a) + r = Q(-a + a) + R;$$

$$\text{or, } -a^3 + ma^2 - na + r = R.$$

EXERCISE XIX.

139. In solving the following examples, the student should prove the work by division, until the principle is understood.

1. If $y^3 - 5y^2 + 9y - 7$ be divided by $y - 2$, what will be the remainder?

Writing 2 in the place of y , we have (137)
 $(2)^3 - 5(2)^2 + 9(2) - 7 = -1$, Ans.

2. If the same quantity be divided by $y + 2$, what will be the remainder?

Writing -2 in the place of y , we have (138)
 $(-2)^3 - 5(-2)^2 + 9(-2) - 7 = -53$, Ans.

3. If $y^3 + a^3$ be divided by $y - a$, what will be the remainder? Ans. (137), $(a)^3 + a^3$, or $2a^3$.

4. If $y^3 + a^3$ be divided by $y + a$, what will be the remainder? Ans. (138), $(-a)^3 + a^3$, or 0.

Find the remainder

5. When $y^2 - 6y + 5$ is divided by $y - 1$.
6. " $y^2 - 6y + 5$ " " $y + 1$.
7. " $y^2 - 2y - 35$ " " $y - 5$.
8. " $y^2 - 2y - 35$ " " $y + 5$.
9. " $x^3 - 6x^2 + 11x - 6$ is divided by $x - 2$.
10. " $x^3 + 6x^2 + 11x + 6$ " " $x + 2$.
11. " $x^5 + 32$ is divided by $x - 2$.
12. " $x^5 + 32$ " " $x + 2$.

Find the remainder:

13. When $y^3 - 3y^2 + 2y$ is divided by $y - 3$.

14. " $y^3 - 3y^2 + 2y$ " " $y + 3$.

One quantity is said to be divisible by another when the remainder is equal to 0.

15. Is $y^2 - a^2$ divisible by $y - a$?

16. " $y^2 - a^2$ " $y + a$?

17. " $x^3 + a^3$ " $x - a$?

18. " $x^3 + a^3$ " $x + a$?

19. " $x^4 + a^4$ " $x + a$, or $x - a$?

20. " $x^4 - a^4$ " $x + a$, or $x - a$?

21. " $y^5 + a^5$ " $y + a$, or $y - a$?

22. " $y^5 - a^5$ " $y + a$, or $y - a$?

23. " $y^6 + a^6$ " $y + a$, or $y - a$?

24. " $y^6 - a^6$ " $y + a$, or $y - a$?

FOUR IMPORTANT CASES.

140. There are some cases in Division which occur so often in algebraic operations that they should be carefully noticed and remembered.

In considering these cases, we shall employ $2n$ to denote an *even* number, and $2n + 1$ to denote an *odd* number. Let it be observed that $(-a)^{2n} = a^{2n}$ (48), and $(-a)^{2n+1} = (-a)^{2n} \times -a = -a^{2n+1}$.

141. CASE I.—*The difference of two equal odd powers of any two numbers is divisible by the difference of the numbers.*

That is, $y^{2n+1} - a^{2n+1}$ is divisible by $y - a$.

For, if we divide the former by the latter, and continue the operation until we obtain a remainder which does not contain y , that remainder will be (137)

$$a^{2n+1} - a^{2n+1} ; \text{ which } = 0.$$

$$\text{Thus, } \frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$$

We may write the quotient by inspection by observing that:

1°. The number of terms in the quotient is equal to the exponent of the powers, and the signs of all the terms are plus.

2°. The first term of the quotient is obtained by dividing the first term of the dividend by the first term of the divisor.

3°. Each succeeding term of the quotient may be found by dividing the preceding term by the first term of the divisor, and multiplying the result by the second term of the divisor (disregarding the sign).

EXERCISE XX.

Write by inspection the results of the following:

1. $(a^3 - 1) \div (a - 1)$. 5. $(n^4 - 16) \div (n - 2)$.
2. $(b^3 - 8) \div (b - 2)$. 6. $(8 - 27x^3) \div (2 - 3x)$.
3. $(a^7 - b^7) \div (a - b)$. 7. $(x^5 - y^5) \div (x - y)$.
4. $(b^4 - x^4) \div (b - x)$. 8. $(32a^5 - 1) \div (2a - 1)$.
9. $(25a^4x^2 - 9) \div (5a^2x - 3)$.
10. $(1 - 64a^6c^9) \div (1 - 4a^2c^3)$.
11. $(216 - 64) \div (6 - 4)$.
12. $(1331 - 343c^4) \div (11 - 7c)$.

142. CASE II. *The sum of two equal odd powers of two numbers is divisible by the sum of the numbers.*

That is, $y^{2n+1} + a^{2n+1}$ is divisible by $y + a$

For (138), the remainder after division is $(-a)^{2n+1} + a^{2n+1}$, or (140) $-a^{2n+1} + a^{2n+1}$, which $= 0$.

$$\text{Thus, } \frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

The quotient may be found as in Case I, but the signs are alternately plus and minus.

Write by inspection the results of the following:

13. $(a^3 + x^3) \div (a + x)$. 17. $(64 + c^6) \div (4 + c^2)$.
 14. $(x^5 + 1) \div (x + 1)$. 18. $(a^9 + 125) \div (a^3 + 5)$.
 15. $(8 + a^3) \div (2 + a)$. 19. $(a^5 + 32b^5) \div (a + 2b)$.
 16. $(a^7 + b^7) \div (a + b)$. 20. $(343 + a^3) \div (7 + a)$.
 21. $(64a^3 + 1000n^6) \div (4a + 10n^2)$.
 22. $(1331 + 216) \div (11 + 6)$.
 23. $(1024a^5 + 243b^5) \div (4a + 3b)$.
 24. $[(a + b)^3 + x^3] \div [(a + b) + x]$.

143. CASE III.—*The difference of two equal even powers of two numbers is divisible by the sum and also by the difference of the numbers.*

That is, $y^{2n} - a^{2n}$ is divisible by $y + a$ and $y - a$.

For (138), the remainder after dividing by $y + a$ is $(-a)^{2n} - a^{2n}$, which $= 0$.

and (137), the remainder after dividing by $(y - a)$ is $(a)^{2n} - (a)^{2n}$, which $= 0$.

When the divisor is the difference of the numbers we find the quotient as in Case I; and when the divisor is the sum of the numbers, we find the quotient as in Case II.

Write by inspection the results of the following:

25. $(a^4 - 1) \div (a - 1)$. 30. $(x^4 - 81) \div (x - 3)$.
 26. $(a^4 - 1) \div (a + 1)$. 31. $(a^4 x^8 - 1) \div (ax^2 - 1)$.
 27. $(a^6 - b^6) \div (a + b)$. 32. $(64 - c^6) \div (2 + c)$.
 28. $(a^6 - b^6) \div (a - b)$. 33. $(64 - c^6) \div (2 - c)$.
 29. $(16x^4 - 1) \div (2x + 1)$. 34. $(1 - a^4 c^{12}) \div (1 + ac^3)$.
 35. $(81x^4 - 16y^4) \div (3x + 2y)$.
 36. $(625a^8 - c^4) \div (5a^2 - c)$.
 37. $(x^2 - 289a^4) \div (x - 17a^2)$.
 38. $(1296a^4 - 16) \div (6a + 2)$.
 39. $\frac{(1+x)^2 - 9}{(1+x) + 3}$ 40. $\frac{(a+2)^4 - 16x^4}{(a+2) - 2x}$
 41. $\frac{(x^2 + 2x)^2 - (3x + 1)^2}{(x^2 + 2x) - (3x + 1)}$
 42. $\frac{a^4 - (3a^2 - 2a + 1)^4}{a + (3a^2 - 2a + 1)}$

144. CASE IV.—*The sum of two equal even powers of two numbers is not divisible by either the sum or the difference of the numbers.*

That is, $y^{2n} + a^{2n}$ is not divisible by $y + a$ or $y - a$.

For (138), the remainder after dividing by $y + a$, is

$$-(-a)^{2n} + (a)^{2n}; \text{ which } = 2a^{2n};$$

and (137), the remainder after dividing by $y - a$, is

$$a^{2n} + a^{2n}; \text{ which } = 2a^{2n}.$$

CHAPTER III.

FACTORIAL PROPERTIES AND RELATION OF NUMBERS.

FACTORING.

145. Factoring is the operation of finding the factors of a number.

FACTORING OF POLYNOMIALS.

146. *When all the terms of a polynomial have a common factor.*

EXERCISE XXI.

1. Find the factors of $6a^2x^2 + 10bx^2y - 2mx^2z$.

$2x^2$ will divide each term. Dividing through by $2x^2$

we have $3a^2 + 5by - mz$. Hence,

$$6a^2x^2 + 10bx^2y - 2mx^2z = (2x^2)(3a^2 + 5by - mz).$$

Rule.—Divide the given polynomial by any monomial that will divide each of its terms; the divisor will be one factor, and the quotient will be the other.

Express the following in terms of their factors:

2. $ax + x^2$.

9. $a^2b + c^2b - db^2$.

3. $x^2y - xy^2$.

10. $6a^2c - 3abc + 7a^2c^2$.

4. $5ab + 10a^2$.

11. $8a^3b^2 + 16a^2b^3 - 32a^4b^4$.

5. $6a^3c - 9a^2c^2$.

12. $14a^3c - 70a^2c^2x + 28a^4c^2$.

6. $12ab^2c + 20bc^2$.

13. $a^3b^2x^3 - a^4bx^4y + a^3bx^3$.

7. $21a^4x^2 - 35a^3x^3$.

14. $3a^2x - 6b^3x^2y - 27xy^2$.

8. $a(m) + b(m)$.

15. $a(a + x) + b(a + x)$

147. *When the terms of a polynomial can be so arranged as to show a common factor.*

16. Find the factors of $x^2 + mx + nx + mn$.

$$\begin{aligned} x^2 + mx + nx + mn &= (x^2 + mx) + (nx + mn). \\ &= x(x + m) + n(x + m). \\ &= (x + n)(x + m). \end{aligned}$$

17. Find the factors of $ab + ay - by - y^2$.

$$\begin{aligned} ab + ay - by - y^2 &= (ab + ay) - (by + y^2). \\ &= a(b + y) - y(b + y). \\ &= (a - y)(b + y). \end{aligned}$$

18. Find the factors of $x^2 + 9x + 20$.

$$\begin{aligned} x^2 + 9x + 20 &= (x^2 + 4x) + (5x + 4 \times 5) \\ &= x(x + 4) + 5(x + 4) \\ &= (x + 5)(x + 4). \end{aligned}$$

Express the following in terms of their factors:

- | | |
|-----------------------------------|-----------------------------|
| 19. $x^2 + ax + cx + ac$. | 27. $x^2 + (a + b)x + ab$. |
| 20. $y^2 + by + ny + bn$. | 28. $x^2 + 7x + 12$. |
| 21. $ab + bm - an - mn$. | 29. $y^2 - (m + n)y + mn$. |
| 22. $ab - bm - an + mn$. | 30. $y^2 - 8y + 15$. |
| 23. $y^2 - ay - by + ab$. | 31. $x^2 + (a - b)x - ab$. |
| 24. $z^2 + cz - bz - bc$. | 32. $x^2 + 3x - 28$. |
| 25. $a^2b^2 + b^2n - a^2m - mn$. | 33. $y^2 - (m - n)y - mn$. |
| 26. $ac^2 - c^2y^2 - ab + by^2$. | 34. $y^2 - 3y - 10$. |

FACTORING OF BINOMIALS.

148. *When the binomial is in the form of two squares with a minus sign between them.*

1. Find the factors of $x^2 - 25$.

The square root of x^2 is x , and the square root of 25 is 5.
Hence (122),

$$x^2 - 25 = (x + 5)(x - 5).$$

Rule.—*Find the square root of the first, and also of the second term. Take the sum of these roots for one factor, and the difference of the roots for the other factor.*

EXERCISE XXII.

Find the factors of:

- | | |
|-------------------------|-------------------------------|
| 2. $x^2 - 81$. | 9. $a^2b^2 - 4$. |
| 3. $x^2 - 49$. | 10. $16a^2 - m^4$. |
| 4. $4y^2 - 25$. | 11. $9a^2b^2 - 1$. |
| 5. $81a^4b^2 - 144$. | 12. $100x^2y^2 - 121a^2b^2$. |
| 6. $121a^6c^2 - 4x^2$. | 13. $64c^6x^2 - 25y^8$. |
| 7. $36x^8 - 169$. | 14. $49a^2y^6 - 9a^6y^2$. |
| 8. $a^4 - 400b^2$. | 15. $1 - 225a^6b^4c^2$. |

149. A binomial may often be resolved into three or more factors. Thus:

$$\begin{aligned}
 a^8 - b^8 &= (a^4 + b^4)(a^4 - b^4) \\
 &= (a^4 + b^4)(a^2 + b^2)(a^2 - b^2) \\
 &= (a^4 + b^4)(a^2 + b^2)(a + b)(a - b).
 \end{aligned}$$

Find all the factors of:

$$16. x^4 - 1. \quad 18. c^4 - 625.$$

$$17. 1 - 16a^8. \quad 19. a^{16} - b^{16}.$$

20. Find the factors of $(a + b)^2 - (c + d)^2$.

$$\begin{aligned}
 (a + b)^2 - (c + d)^2 &= \{(a + b) + (c + d)\} \{(a + b) - (c + d)\} \\
 &= \{a + b + c + d\} \{a + b - c - d\}
 \end{aligned}$$

Find the factors of:

- | | |
|--------------------------|--|
| 21. $(x + a)^2 - n^2$. | 26. $(x + a)^2 - (x - a)^2$. |
| 22. $(x - b)^2 - m^2$. | 27. $(x^2 + 2x)^2 - (2x + 1)^2$. |
| 23. $(x + 2)^2 - 25$. | 28. $(x^3 + 4x^2)^2 - (3x^2 + 2x)^2$. |
| 24. $36 - (y - 6)^2$. | 29. $(x^3 - 6x)^2 - (x^2 - 6)^2$. |
| 25. $9x^2 - (x + 5)^2$. | 30. $x^2 - 2ax + a^2 - 9b^2$. |

150. *When the binomial is of the form of equal powers of any two numbers with a minus sign between them.*

31. Find the factors of $a^5 - b^5$.

Dividing by $a - b$ we get $a^4 + a^3b + a^2b^2 + ab^3 + b^4$.

Hence, $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

Rule—Divide by the difference of the two numbers; the divisor will be one factor, and the quotient the other.

Find the factors of:

- | | | |
|--------------------------|-----------------------------|----------------------------------|
| 32. $a^3 - b^3$. | 36. $1 - x^4$. | 40. $125x^3 - 8y^3$. |
| 33. $x^3 - 2^3$. | 37. $125 - a^3$. | 41. $(x + 1)^4 - 16$. |
| 34. $x^4 - 3^4$. | 38. $16a^4 - b^4$. | 42. $(x+a)^3 - (x-a)^3$. |
| 35. $y^3 - 64$. | 39. $8x^3 - 27y^3$. | 43. $729a^3 - 1331b^3$. |

151. *When the binomial is of the form of equal odd powers of any two numbers with the sign plus between them.*

44. Find the factors of $x^3 + y^3$.

Dividing by $x + y$, we get $x^2 - xy + y^2$ (**142**). Hence,

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

Rule.—Divide by the sum of the two numbers; the divisor will be one factor and the quotient will be the other.

Find the factors of:

- | | | |
|--------------------------|----------------------------|--------------------------------|
| 45. $x^3 + 2^3$. | 48. $8x^3 + 1$. | 51. $125a^3 + 64b^3$. |
| 46. $x^5 + 3^5$. | 49. $a^5 + 32$. | 52. $(x + 1)^3 + 27$. |
| 47. $x^3 + 64$. | 50. $64b^3 + 125$. | 53. $216 + (1 + a)^3$. |

FACTORING OF TRINOMIALS.

152. *When the trinomial may be factored by reversing the operations of Arts. 113 to 116.*

EXERCISE XXIII.

1. Find the factors of $x^2 + 11x + 30$.

The first term of each binomial factor is x , $\therefore x \times x = x^2$ (113), and the second terms are $+5$ and $+6$, $\therefore +5 + 6 = +11$, and $+5 \times +6 = 30$. Hence,

$$x^2 + 11x + 30 = (x + 5)(x + 6).$$

2. Find the factors of $4a^4c^2 - 14ab^3c + 10b^6$.

The square root of $4a^4c^2$ is $2a^2c$, which is the first term of each binomial factor. $-14ab^3c \div 2a^2c = -7b^3$. Now we wish to find two numbers whose sum is $-7b^3$ and whose product is $10b^6$, which are evidently $-5b^3$ and $-2b^3$, and these are the second terms of the factors. Hence,

$$4a^4c^2 - 14ab^3c + 10b^6 = (2a^2c - 5b^3)(2a^2c - 2b^3).$$

Rule I.—*Extract the square root of the first term, for the first term of each of the factors.*

II. *For the second terms of the factors, take the two numbers whose sum is the quotient obtained by dividing the second term of the trinomial by the first term of the factors, and whose product is the third term of the trinomial.*

Express the following in terms of their factors:

3. $x^2 + 9x + 18$.

8. $x^2 + 12ax + 32a^2$.

4. $x^2 + 9x + 14$.

9. $x^2 + 14nx + 48n^2$.

5. $x^2 + 9x + 8$.

10. $x^4 + 10cx^2 + 16c^2$.

6. $x^2 + 9x + 20$.

11. $y^4 + 16a^2y^2 + 64a^4$.

7. $y^2 + 10y + 21$.

12. $y^6 + 12cy^3 + 27c^2$.

Express the following in terms of their factors :

- | | |
|-----------------------|-----------------------------------|
| 13. $y^2 + 13y + 42.$ | 16. $z^6 + 4a^2z^3 + 4a^4.$ |
| 14. $a^2 + 18a + 72.$ | 17. $a^2b^2 + 5abn + 6n^2.$ |
| 15. $n^2 + 20n + 75.$ | 18. $a^4b^2 + 11a^2bc^2 + 18c^4.$ |
| 19. $x^2 - 13x + 30.$ | |

The second terms of the binomial factors in this example are -10 and -3 , $\therefore (-10) + (-3) = -13$, and $-10 \times -3 = +30$. Hence :

$$x^2 - 13x + 30 = (x - 10)(x - 3).$$

- | | |
|------------------------|-----------------------------------|
| 20. $x^2 - 11x + 10.$ | 28. $x^2 - 16ax + 63a^2.$ |
| 21. $x^2 - 11x + 18.$ | 29. $x^2 - 14nx + 45n^2.$ |
| 22. $x^2 - 11x + 24.$ | 30. $x^4 - 10cx^2 + 21c^2.$ |
| 23. $x^2 - 11x + 28.$ | 31. $y^4 - 16a^2y^2 + 48a^4.$ |
| 24. $y^2 - 15y + 56.$ | 32. $y^6 - 12by^3 + 35b^2.$ |
| 25. $y^2 - 17y + 72.$ | 33. $n^6 - 8a^2n^3 + 15a^4.$ |
| 26. $a^2 - 15a + 54.$ | 34. $a^2x^2 - 10aux + 24c^2.$ |
| 27. $u^2 - 21n + 108.$ | 35. $x^6y^4 - 13x^3y^2n + 40n^2.$ |
| 36. $x^2 + 3x - 40.$ | |

The second terms of the binomial factors in this example are $+8$ and -5 , $\therefore +8 + (-5) = +3$, and $+8 \times -5 = -40$. Hence (115).

$$x^2 + 3x - 40 = (x + 8)(x - 5).$$

NOTE.—When the last term of the trinomial is —, the last terms of the binomial factors have contrary signs, and the greater has the sign of the second term of the trinomial.

- | | |
|------------------------|--------------------------------|
| 37. $x^2 + 5x - 14.$ | 45. $x^2 + 7ax - 120a^2.$ |
| 38. $x^2 + 5x - 24.$ | 46. $x^2 + 3nx - 130n^2.$ |
| 39. $x^2 + 5x - 36.$ | 47. $x^4 + 5cx^2 - 300c^2.$ |
| 40. $x^2 + 7x - 44.$ | 48. $y^4 + 7a^2y^2 - 170a^4.$ |
| 41. $x^2 + 9x - 36.$ | 49. $y^6 + 7cy^3 - 60c^2.$ |
| 42. $x^2 + 9x - 90.$ | 50. $z^6 + 4m^3z^3 - 96m^6.$ |
| 43. $a^2 + 11a - 26.$ | 51. $a^2x^2 + 6abx - 40b^2.$ |
| 44. $n^2 + 15n - 100.$ | 52. $x^4y^2 + 8cx^2y - 48c^2.$ |

$$53. x^2 - 4x - 77.$$

The second terms of the binomial factors in this example are -11 and $+7$. See Art. 116. Hence,

$$x^2 - 4x - 77 = (x - 11)(x + 7).$$

$$54. x^2 - 5x - 14.$$

$$62. x^2 - 7ax - 120a^2.$$

$$55. x^2 - 5x - 24.$$

$$63. x^2 - 5xn - 104n^2.$$

$$56. x^2 - 6x - 27.$$

$$64. x^4 - 6ax^2 - 40a^2.$$

$$57. x^2 - 10x - 39.$$

$$65. a^4 - 6a^2b - 72b^2.$$

$$58. x^2 - 10x - 75.$$

$$66. y^6 - any^3 - 90a^2n^2.$$

$$59. x^2 - 9x - 52.$$

$$67. z^6 - 2a^2z^3 - 35a^4.$$

$$60. c^2 - 11c - 26.$$

$$68. a^2x^2 - 2abx - 3b^2.$$

$$61. m^2 - 20m - 125.$$

$$69. y^6z^8 - y^3z^4 - 110.$$

153. *When the trinomial is a perfect square.*

This is but a particular form of the preceding case; but, on account of its importance, it demands special attention.

EXERCISE XXIV.

1. Find the factors of $x^2 + 12x + 36$.

The second terms of the binomial factors are $+6$ and $+6$, or $\frac{1}{2}$ of $+12$; $\therefore +6 + 6 = +12$, and $+6 \times +6 = +36$. Hence,

$$x^2 + 12x + 36 = (x + 6)(x + 6) = (x + 6)^2.$$

2. Find the factors of $a^2x^2 - 6abx + 9b^2$.

The square root of a^2x^2 is ax ; $-6abx \div 2$ times $ax = -3b$, which is the second term of each of the binomial factors. Hence,

$$a^2x^2 - 6abx + 9b^2 = (ax - 3b)(ax - 3b) = (ax - 3b)^2.$$

NOTE.—It should be observed that the absolute value of each of the second terms of the binomial factors is the square root of the third term of the given trinomial.

Find the factors of:

- | | |
|-------------------------------|--------------------------------------|
| 3. $x^2 + 14x + 49.$ | 13. $1 - 6ab^3 + 9a^2b^6.$ |
| 4. $x^2 - 16x + 64.$ | 14. $49a^2 - 112ab + 64b^2.$ |
| 5. $y^2 - 24y + 144.$ | 15. $16x^2y^6 - 8xy^4z^2 + y^2z^4.$ |
| 6. $z^2 + 20z + 100.$ | 16. $121x^6 - 286x^3y + 169y^2.$ |
| 7. $x^2 + 30x + 225.$ | 17. $4z^6 - 63z^3 + 289.$ |
| 8. $x^2 - 6ax + 9a^2.$ | 18. $9m^2n^2 + 24amn + 16a^2.$ |
| 9. $x^2 + 12ax + 36a^2.$ | 19. $4b^2x^2 + 12bx^2y + 9x^2y^2.$ |
| 10. $x^2 - 18c^2x + 81c^4.$ | 20. $324a^4c^2 - 36a^3c^4 + a^2c^6.$ |
| 11. $a^3 - 2a^2b + b^2.$ | 21. $x^2z^4 + 38xz^5 + 361z^6.$ |
| 12. $a^6 + 10a^3c^2 + 25c^4.$ | 22. $64n^6 + 160an^3 + 100a^2.$ |

154. *When the first term of the trinomial is not a perfect square.*

EXERCISE XXV.

1. Find the factors of $6x^2 + 5x - 4$.

The first terms of the two factors might be $6x$ and x , or $2x$ and $3x$, since $6x \times x = 6x^2$, and $2x \times 3x = 6x^2$.

Likewise the last terms of the two factors might be 4 and 1 , or 2 and 2 (disregarding the signs).

$$\begin{array}{r} 3x + 4 \\ 2x - 1 \\ \hline 8x \\ - 3x \\ \hline + 5x \end{array}$$

From these it is necessary to select such as will produce the middle term of the trinomial, and they are found by trial to be $3x$ and $2x$, and $+4$ and -1 .

$$\therefore 6x^2 + 5x - 4 = (3x + 4)(2x - 1).$$

2. Find the factors of $18x^2 - 9x - 5$.

155. The following method of solution is often preferable, which, for the present, is given without demonstration.

Dividing the first term and multiplying the last term by 18, the coefficient of x^2 ,

Factoring (152),

$$x^2 - 9x - 90.$$

$$(x - 15)(x + 6).$$

Multiplying x in each factor by
18, the coefficient of x^2 ,

Dividing by 18, or 6×3 ,

or

$$\begin{array}{r} (18x - 15) (18x + 6). \\ \hline (18x - 15) (18x + 6), \\ \hline \quad 3 \qquad \quad 6 \\ (6x - 5) (3x + 1). \end{array}$$

Find the factors of:

- | | |
|-----------------------------|------------------------------|
| 3. $6x^2 + 11x + 3$. | 4. $12x^2 - 17x - 40$. |
| 5. $10x^2 - 11x + 3$. | 6. $20x^2 - 41x + 20$. |
| 7. $12x^2 + 17x - 7$. | 8. $56x^2 - 11x - 15$. |
| 9. $18x^2 - 3x - 10$. | 10. $24x^2 - 14x - 5$. |
| 11. $12x^2 + 23x + 10$. | 12. $24x^2 + x - 3$. |
| 13. $6x^2 + 13ax + 6a^2$. | 14. $24x^2 - 22x + 5$. |
| 15. $8a^2 + 14ab - 15b^2$. | 16. $24x^2 + x - 25$. |
| 17. $11a^2 - 23ac + 2c^2$. | 18. $12a^4 + a^2b^2 - b^4$. |

156. *When the trinomial is the difference of two perfect squares.*

157. A trinomial is a perfect square when the middle term is twice the product of the square roots of the first and last terms.

Thus, $x^4 + x^2y^2 + y^4$ is not a perfect square, but $x^4 + 2x^2y^2 + y^4$ is, and its square root is $x^2 + y^2$. Hence, $x^4 + x^2y^2 + y^4$ is the difference between two perfect squares. For,

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= (x^2 + y^2)^2 - (xy)^2. \end{aligned}$$

EXERCISE XXVI.

1. Find the factors of $4a^4 + 11a^2b^4 + 25b^8$.

$$\begin{aligned} 4a^4 + 11a^2b^4 + 25b^8 &= 4a^4 + 20a^2b^4 + 25b^8 - 9a^2b^4. \\ &= (2a^2 + 5b^4)^2 - (3ab^2)^2. \\ &= (2a^2 + 3ab^2 + 5b^4)(2a^2 - 3ab^2 + 5b^4). \end{aligned}$$

Find the factors of:

- | | |
|---------------------------|----------------------------------|
| 2. $a^4 + a^2 + 1$. | 3. $9a^4 + 21a^2c^2 + 25c^4$. |
| 4. $a^4 + a^2b^2 + b^4$. | 5. $49a^4 - 15a^2b^2 + 121b^4$. |
| 6. $1 + 2x^2 + 9x^4$. | 7. $16x^4 - 65x^2y^2 + 49y^4$. |
| 8. $25a^4 + a^2 + 1$. | 9. $25x^4 - 41x^2y^2 + 16y^4$. |
| 10. $16x^4 - 17x^2 + 1$. | 11. $100a^8 + 16a^4 + 1$. |
| 12. $81 - 34y^2 + y^4$. | 13. $144 - 217x^2 + 81x^4$. |

MISCELLANEOUS EXAMPLES.

EXERCISE XXVII.

158. In the following, first take out the common monomial factor if any, then apply the preceding principles.

- | | |
|----------------------------|-----------------------------------|
| 1. $x^2 - 1$. | 2. $x^2 - 15x + 56$. |
| 3. $x^3 + 1$. | 4. $x^2 + 23x + 60$. |
| 5. $x^4 - 1$. | 6. $x^2 - 4x - 77$. |
| 7. $x^3 - 27$. | 8. $x^2 + 5x - 104$. |
| 9. $x^3 + 27$. | 10. $3x^2 + 4x - 4$. |
| 11. $16x^4 - a^4$. | 12. $6x^2 + 5x - 4$. |
| 13. $125 + a^3$. | 14. $4x^2 + 13x + 3$. |
| 15. $x^4 + a^3x$. | 16. $6a^2 - 19a + 10$. |
| 17. $a^5 - a^2b^3$. | 18. $x^3 + x^4 + 1$. |
| 19. $5 + 40x^3$. | 20. $4x^4 + 3x^2 + 1$. |
| 21. $56x^3 - 189$. | 22. $9a^4 - 10a^2 + 1$. |
| 23. $64x^7 - x$. | 24. $25 - 19x^2 + x^4$. |
| 25. $ax^4 + ax^2 + a$. | 26. $5x^3 - x^2 + 5x - 1$. |
| 27. $3x^4 - 39x^2 + 120$. | 28. $6a^2b^2 - ab^3 - 12b^4$. |
| 29. $x^3 + 2x^2 - 15x$. | 30. $81a^4 + 23a^2b^2 + 16b^4$. |
| 31. $2x^2 - 30x + 112$. | 32. $49a^4 + 38a^2x^2 + 9x^4$. |
| 33. $5x^8 - 25x^6 - 420$. | 34. $25x^6 - 9x^3c^2 + 16c^4$. |
| 35. $3x^6 + 15x^3 - 252$. | 36. $px^2 - qx^2 - py^2 + qy^2$. |

37. $7x^4 - 49x^2 - 420$. 38. $x^3 - x^2 + x - 1$.
 39. $ax^4 - 6ax^2 - 40a$. 40. $x^4 - 3x^3 - x^2 + 3x$.
 41. $24x^3 + 34x^2 - 80x$. 42. $4a^2 + 8ab - 3ac - 6bc$.
 43. $24a^3 - 34a^2 - 80a$. 44. $(x^2 + 2x)^2 - (2x + 1)^2$.
 45. $a^4x^2 - 5a^2x - 24$. 46. $x^3 + ax^2 - a^2x - a^3$.
 47. $a^7 + a^5 + a^3$. 48. $4a^2b^2 - (a^2 + b^2 - c^2)^2$.
 49. $3x^2y^2 - xy^3 - 2y^4$. 50. $(x^2 - y^2 - z^2)^2 - 4y^2z^2$.

GREATEST COMMON DIVISOR.

159. The **Factors** of a number are sometimes called its **Divisors**, since it may be divided by any one of them.

160. A **Common Divisor** of two or more numbers is any factor common to those numbers.

161. The **Greatest Common Divisor** of two or more numbers, denoted by **G. C. D.**, is the greatest factor common to those numbers.

162. *When the numbers can be resolved into factors by inspection.*

EXERCISE XXVIII.

1. Find the **G. C. D.** of:

$$28a^3b^4, \quad 42a^5b^3, \quad \text{and} \quad 70a^2b^5c.$$

$$28a^3b^4 = 2^2 \times 7 \times a^3 \times b^4$$

$$42a^5b^3 = 2 \times 3 \times 7 \times a^5 \times b^3$$

$$70a^2b^5c = 2 \times 5 \times 7 \times a^2 \times b^5 \times c.$$

$$\therefore \text{the G. C. D.} = 2 \times 7 \times a^2 \times b^3 = 14a^2b^3.$$

2. Find the **G. C. D.** of:

$$6a(x^2 + 8x + 15), \quad 8a(x^2 - 2x - 15), \quad 4a^2(x^2 - 9),$$

$$6a(x^2 + 8x + 15) = 2 \times 3 \times a \times (x + 3)(x + 5)$$

$$8a(x^2 - 2x - 15) = 2^3 \times a \times (x + 3)(x - 5)$$

$$4a^2(x^2 - 9) = 2^2 \times a^2 \times (x + 3)(x - 3)$$

$$\therefore \text{the G. C. D.} = 2 \times a \times (x + 3) = 2ax + 6a.$$

Hence, Rule: 1°, *Resolve each number into its lowest factors.* 2°, *Select from these the lowest power of each common factor, and find the product of these powers.*

Find the G. C. D. of:

3. $4a^3b$ and $6a^2b^2x$.
4. a^4b^2c and b^3c^2x .
5. $12b^2xy$ and $8ab^3x^2$.
6. $5d^3n^2$ and $7a^2bn$.
7. $18c^3n^2x$ and $12b^2c^4x$.
8. $52a^2b^2c$ and $117bc^2x$.
9. $63c^4x^3z^2$ and $91a^2x^2z^3$.
10. $a(a+b)^3$ and $a^2(a+b)^2$.
11. $8a^3b^2c^4$, $16a^3b^3c^3$, and $20a^4b^4c^2$.
12. $40x^4y^3c^2$, $64x^3y^5c^4$, and $120x^2yc^3$.
13. $(x+y)^2$ and $x^2 - y^2$.
14. $4x^2 - 1$ and $(2x-1)^2$.
15. $(a-b)^2$ and $a^2 - b^2$.
16. $a^4 - b^4$ and $a^2 - b^2$.
17. $x^3 - c^3$ and $x - c$.
18. $4(x-y)^4$ and $6(x-y)^3$.
19. $x^2 + 9x + 20$ and $x^2 - 16$.
20. $x^2 - 9x + 20$ and $x^2 - 25$.
21. $x^2 + 3x - 54$ and $x^2 - 81$.
22. $x^2 - 4x - 60$ and $x^2 - 36$.
23. $x^2 + 7x - 18$ and $x^2 + 12x + 27$.
24. $x^2 + 6x - 16$ and $x^2 - 6x + 8$.
25. $x^2 - 2x - 35$, $x^2 - 9x + 14$, and $x^2 - 49$.
26. $x^2 + 11x - 26$, $x^2 + 9x - 52$, and $x^2 + 15x + 26$.

163. When one of the numbers can be factored and the other cannot, we may ascertain by trial which of the factors of the first are contained in the second.

27. $x^2 - 7x + 10$ and $x^3 - 6x^2 + 11x - 6$.

$x^2 - 7x + 10 = (x-2)(x-5)$. Now by trial we find that $x-2$ will divide $x^3 - 6x^2 + 11x - 6$, and $x-5$ will not. Hence, $x-2$ is the G. C. D.

28. $x^2 - y^2$ and $x^3 + y^3$.

29. $x^2 + x - 6$ and $2x^3 - 12x^2 + 21x - 10$.

30. $a^2 - 7ab + 12b^2$ and $5a^3 - 18a^2b + 11ab^2 - 6b^3$.

164. *When the numbers can not be readily resolved into their factors by inspection.*

31. Find the G. C. D. of $70p$ and $165p$, where p may be regarded as representing a polynomial.

The G. C. D. in this case is evidently $5p$, which we obtain by inspection; but we shall find this G. C. D. by the two following methods, for the purpose of illustrating three important principles.

1°. We may divide the greater number by the less, then divide the less number by the remainder, and continue to divide the last divisor by the last remainder until nothing remains; the last divisor, as $5p$, will be the G. C. D. required.

FIRST SOLUTION.

$$\begin{array}{r}
 70p \overline{) 165p} (2 \\
 \underline{140p} \\
 25p \overline{) 70p} (2 \\
 \underline{50p} \\
 20p \overline{) 25p} (1 \\
 \underline{20p} \\
 5p \overline{) 20p} (4 \\
 \underline{20p} \\
 0
 \end{array}$$

2°. We may divide both numbers by any common factor as 5, and set that factor aside for one of the factors of the G. C. D.

3°. We may multiply or divide either dividend or divisor by any factor which is not a factor of the other.

Thus, we divide $14p$ by 2, because 2 is not a factor of $33p$; again we divide $33p$ by 3, because 3 is not a factor of $14p$.

ABBREVIATED METHOD.

$$\begin{array}{r}
 5 \mid 70p, 165p. \\
 \underline{14p, 33p.} \\
 14p \div 2 = 7p; 33p \div 3 = 11p \\
 7p \overline{) 11p} (1 \\
 \underline{7p} \\
 4p \overline{) 7p} (1 \\
 \underline{4p} \\
 3p \overline{) 4p} (1 \\
 \underline{3p} \\
 p \overline{) 3p} (3 \\
 \underline{3p}
 \end{array}$$

165. DEMONSTRATION OF THE 1^o PRINCIPLE.

Evidently, $165p - 2(70p) = 25p$.

That is, $165p$, and $-2(70p)$ are the comp. parts of $25p$; hence, any number which will divide each of the two former will divide the latter (130).

In general, any number which will divide the dividend and divisor will also divide the divisor and remainder.

Hence, it follows that any number:

which will divide $165p$ and $70p$, will divide $70p$ and $25p$;
 which will divide $70p$ and $25p$, will divide $25p$ and $10p$;
 which will divide $25p$ and $10p$, will divide $10p$ and $5p$.

Now $5p$ is the G. C. D. of $10p$ and $5p$, hence it is the G. C. D. of $165p$ and $70p$.

The same reasoning may be applied to any two numbers whatever.

32. Find the G. C. D. of

$$x^3 + 7x^2 + 17x + 14 \text{ and } x^2 - x - 6.$$

$$x^3 + 7x^2 + 17x + 14 \quad | \quad x^2 - x - 6.$$

$$x^3 - x^2 - 6x \qquad \qquad \qquad x + 8$$

$$\hline 8x^2 + 23x + 14$$

$$8x^2 - 8x - 48$$

$$\hline 31x + 62 = 31(x + 2).$$

Rejecting 31 by Princ. 3^o, $x + 2$ becomes our next divisor.

$$x^2 - x - 6 \quad | \quad x + 2$$

$$x^2 + 2x \quad | \quad x - 3$$

$$\hline -3x - 6$$

$$\hline -3x - 6$$

Hence, $x + 2$ is the G. C. D.

33. Find the G. C. D. of $x^3 + y^3$ and $x^2 - y^2$.

$$(x^3 + y^3) \div (x^2 - y^2) = x \text{ (quo.) and } xy^2 + y^3 \text{ or } y^2 \\ (x + y) \text{ (rem.)}$$

$$\text{Rejecting } y^2, \text{ Princ. 3}^o, (x^2 - y^2) \div (x + y) = x - y.$$

Hence, $x + y$ is the G. C. D.

All the principles involved in finding the G. C. D. are illustrated in the solution of the next example.

34. Find the G. C. D. of $4x^5 + 9x^4 + 2x^3 - 2x^2 - 4x$ and $6x^4 + 10x^3 - 2x^2 + 4x$

Since the factor x is common to both, we divide out by x , and retain the x as a factor of the G. C. D. Again, since 2 is a factor of the second number and not of the first, we divide the second by 2 and throw it away:

The two numbers thus become

$$4x^4 + 9x^3 + 2x^2 - 2x - 4 \text{ and } 3x^3 + 5x^2 - x + 2.$$

Now, in order that the first may be divisible by the second, we multiply it by 3, and then proceed thus:

$$\begin{array}{r} 12x^4 + 27x^3 + 6x^2 - 6x - 12 \quad | \quad 3x^3 + 5x^2 - x + 2. \\ 12x^4 + 20x^3 - 4x^2 + 8x \\ \hline 7x^3 + 10x^2 - 14x - 12. \end{array}$$

We multiply this remainder by 3 to make it divisible by the divisor and continue thus:

$$\begin{array}{r} 21x^3 + 30x^2 - 42x - 36 \\ 21x^3 + 35x^2 - 7x + 14 \\ \hline - 5x^2 - 35x - 50 \\ \text{or } - 5(x^2 + 7x + 10) \end{array}$$

Now rejecting the factor -5 , we continue thus:

$$\begin{array}{r} 3x^3 + 5x^2 - x + 2 \quad | \quad x^2 + 7x + 10 \\ 3x^3 + 21x^2 + 30x \quad 3x - 16 \\ \hline - 16x^2 - 31x + 2 \\ - 16x^2 - 112x - 160 \\ \hline + 81x + 162 \\ \text{or } 81(x + 2) \end{array}$$

Rejecting the factor 81, we proceed thus:

$$\begin{array}{r} x^2 + 7x + 10 \quad | \quad x + 2 \\ x^2 + 2x \\ \hline 5x + 10 \\ 5x + 10 \end{array}$$

Hence, $x(x + 2)$ or $x^2 + 2x$ is the G. C. D.

Find the **G. C. D.** of:

35. $x^2 + 20x + 91$ and $x^2 - 10x - 119$.
36. $x^2 + 28x + 187$ and $x^2 - 2x - 143$.
37. $3x^2 - 6x + 3$ and $6x^2 + 6x - 12$.
38. $10x^2 + x - 60$ and $6x^2 + x - 35$.
39. $28x^2 - 19x + 3$ and $20x^2 + 7x - 3$.
40. $a^3 + 3a^2b + 3ab^2 + b^3$ and $a^3 + b^3$.
41. $3x^3 + 6x^2 - 24x$ and $6x^3 - 96x$.
42. $x^2 - a^2 + 2xy + y^2$ and $x^2 + a^2 + 2ax - y^2$.
43. $y^3 - 5y^2z - 4yz^2 + 2z^3$ and $7y^2 + 10yz + 3z^2$.
44. $7a^2 - 23ab + 6b^2$ and $5a^3 - 18a^2b + 11ab^2 - 6b^3$.
45. $x^4 - 8x^3 + 21x^2 - 20x + 4$ and $2x^3 - 12x^2 + 21x - 10$.
46. $2x^4 - 12x^3 + 19x^2 - 6x + 9$ and $4x^3 - 18x^2 + 19x - 3$.
47. $6x^4 + x^3 - x$ and $4x^3 - 6x^2 - 4x + 3$.
48. $2x^4 + 11x^3 - 13x^2 - 99x - 45$ and $2x^3 - 7x^2 - 46x - 21$.
49. $3x^5 - 10x^3 + 15x + 8$ and $x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6$.

LEAST COMMON MULTIPLE.

166. A **Multiple** of a number is a number of which the given number is a factor.

167. A **Common Multiple** of two or more numbers is a number of which each of the given numbers is a factor.

168. The **Least Common Multiple** of two or more numbers, denoted by **L. C. M.**, is the least number of which each of the given numbers is a factor.

169. Principle:—*The L. C. M. of two or more numbers is the least number that contains the complementary factors of those numbers.*

170. *When the numbers can be resolved into their factors by inspection.*

EXERCISE XXIX.

1. Find the L. C. M. of $15a^2b^3$ and $20ab^4c$.

$$15a^2b^3 = 3 \times 5 \times a^2 \times b^3$$

$$20ab^4c = 2^2 \times 5 \times a \times b^4 \times c.$$

$$\therefore \text{L. C. M.} = 2^2 \times 3 \times 5 \times a^2 \times b^4 \times c = 60a^2b^4c.$$

2. Find the L. C. M. of

$8(x^2 - 9)$, $9(x^2 + 6x + 9)$, and $5(x^2 - 6x + 9)$.

$$8(x^2 - 9) = 2^3(x + 3)(x - 3).$$

$$9(x^2 + 6x + 9) = 3^2(x + 3)^2; \quad 5(x^2 - 6x + 9) = 5(x - 3)^2.$$

$$\therefore \text{L. C. M.} = 2^3 \times 3^2 \times 5 \times (x + 3)^2 (x - 3)^2 = 360(x^4 - 18x^2 + 81).$$

Hence, Rule. 1^o, *Separate each number into its lowest factors.* 2^o, *Select the highest power of each factor and find the product of these powers.*

Find the L. C. M. of:

3. $6a^3b$ and $8a^2b^2c$.

4. a^4b^2c and b^3c^2x .

5. $12c^2x^3$ and $5x^2y^2$.

6. $a^2b^5c^3$ and $a^3b^2c^4$.

7. $10b^3xy$ and $15b^2x^3y$.

8. $a^2(a + b)^2$ and $a(a + b)^3$

9. $16a^4b^2c$, $24a^3b^3c^2$, and $12a^2bc^5$.

10. $25n^3mx$, $9n^2x^4y$, and m^5x^3 .

11. $x^2 + 3x + 2$ and $x^2 - 4$.

12. $a^2 - b^2$ and $(a - b)^2$.

13. $x^2 - 5x + 6$ and $x^2 - 9$.

14. $9 - 4x^2$ and $(3 + 2x)^2$.

15. $x^2 + 8x + 16$ and $x^2 - 16$.
16. $6(x-1)^2$ and $9(x-1)^3$.
17. $x^2 - 9$, $x^2 + 2x - 15$ and $x^2 - 4x - 21$.
18. $x^2 + x - 30$, $x^2 + 12x + 36$ and $x^2 - 10x + 25$.
19. $5(a+b)^2(a-b)$, $6(a+b)^3(a-b)^2$ and $3(a+b)(a-b)^3$.
20. $4(a+b+c)^2(a+b)$, and $6a(a+b+c)(a+b)^3(a-b)$.

171. *When the numbers cannot be easily factored by inspection.*

21. Find the L. C. M. of a^3b^2c and a^2bc^3x .

The L. C. M. in this case is $a^3b^2c^3x$, which we obtain by inspection. For the purpose of illustrating a principle let us solve the problem thus:

The product of the numbers is $a^5b^3c^4x$.

The G. C. D. of the numbers is a^2bc .

$$a^5b^3c^4x \div a^2bc = a^3b^2c^3x, \text{ Ans.}$$

172. Principle. *The L. C. M. of two numbers is equal to their product divided by their G. C. D. Hence, Rule. 1^o. Find the G. C. D. of the two numbers. 2^o. Divide one of the numbers by their G. C. D., and multiply the other by the quotient.*

Find the L. C. M. of:

22. $x^2 - 23x + 112$ and $x^2 + x - 56$.
23. $x^2 + 28x + 187$ and $x^2 - 2x - 143$.
24. $a^2 - 2ab + b^2$ and $a^4 - b^4$.
25. $x^3 + 6x^2 + 11x + 6$ and $x^3 + 7x^2 + 14x + 8$.
26. $x^4 + 13x^2 + 36$ and $x^4 - 13x^2 + 36$.
27. $x^3 + 2x^2y - xy^2 - 2y^3$ and $x^3 - 2x^2y - xy^2 + 2y^3$.

CHAPTER IV.

FRACTIONS.

DEFINITIONS.

173. A Common Fraction is one number written over another with a line between them.

Thus, $\frac{3}{10}$, $\frac{a}{b}$, $\frac{a+b}{a-c}$; which are read 3 10ths, a bths, $(a+b)$ $(a-c)$ ths.

174. The Terms of a fraction are the two numbers by which the fraction is expressed; the Numerator is the term above the line, the Denominator is the term below the line.

175. A Proper Fraction is one whose numerator is less than the denominator; and an **Improper Fraction** is one whose numerator is equal to or exceeds the denominator.

176. An Entire Number or Integer is a number which does not contain a fraction; and a **Mixed Number** is one which contains both an entire and a fractional part.

177. A Complex Fraction is one having a fraction in either of its terms; and a **Compound Fraction** is a fraction of a fraction.

FUNDAMENTAL PRINCIPLES.

178. Every number, whether entire or fractional, may be regarded as representing *a single thing*, or the *coefficient of a single thing*.

Thus, as a single thing, 5 means *one* 5; and $\frac{1}{3}$, *one* $\frac{1}{3}$. As a coefficient of a single thing, 5 means 5 *ones*; and $\frac{1}{3}$, $\frac{1}{3}$ of *one*.

CASE I.

179. A fraction, regarded as a **single thing**, is a **quotient**, the *numerator* being the *dividend*, and the *denominator* the *divisor*. Hence,

1°. The value of a fraction is the quotient obtained by dividing the numerator by the denominator.

Thus, the value of $\frac{12}{3}$ is 4; of $\frac{a^2 - x^2}{a + x}$, $a - x$.

2°. The dividing line has the force of a vinculum or parenthesis, and the sign before it is the sign of the fraction.

3°. The 1°, 2° and 3° principles of Art. 126, are applicable to fractions, if numerator be substituted for dividend, denominator for divisor, and fraction for quotient. Thus,

$$1^\circ. \quad \frac{12}{6} = 2; \text{ but } \frac{12 \times 3}{6}, \text{ or } \frac{12}{6 \div 3} = 2 \times 3 = 6.$$

$$2^\circ. \quad \frac{12}{6} = 2; \text{ but } \frac{12 \div 2}{6}, \text{ or } \frac{12}{6 \times 2} = 2 \div 2 = 1.$$

$$3^\circ. \quad \frac{16}{8} = \frac{16 \times 4}{8 \times 4} = \frac{16 \div 4}{8 \div 4} = 2$$

5°. If the signs of the terms are alike the fraction is plus; if unlike, minus.

$$\text{Thus, } \frac{+8}{+4} \text{ or } \frac{-8}{-4} = +2 \text{ and } \frac{+8}{-4} \text{ or } \frac{-8}{+4} = -2.$$

6°. Changing the sign of either term changes the sign of the fraction, but changing the signs of both terms does not change the sign of the fraction.

CASE II.

180. A fraction, regarded as a co-efficient, represents one or more of the equal parts of unity.

For (179), $\frac{a}{b} = a \div b = a \text{ times } \frac{1}{b}$, Hence,

$$\frac{a}{b} \text{ of one} = a \text{ times } 1 \text{ bth of one}$$

Hence, the denominator shows into how many equal parts the unit is divided, and the numerator shows how many of these parts are taken.

NOTE.—For practical and business purposes, it is perhaps better to consider a fraction as expressing one or more of the equal parts of unity; hence it is so regarded and treated in the Arithmetics of this series; but for the more general purposes of algebra, it is better to treat fractions as quotients.

REDUCTION OF FRACTIONS.

181. Reduction of fractions is changing their terms without altering the value of the fractions.

182. Principle.—*Multiplying a quantity by a number and then dividing the result by the same number, does not change the value of the quantity.*

$$\text{Thus, } \frac{3}{4} \times 12 = 9; 9 \div 12 = \frac{9}{12} \therefore \frac{3}{4} = \frac{9}{12}$$

Hence, the value of a fraction is not changed by multiplying or dividing both of its terms by the same number (179).

To employ the preceding principle in the reduction of fractions, it is necessary first to ascertain by what, and how, to multiply a fraction in order that the product may be an entire number.

183. If a fraction be multiplied by a multiple of the denominator, the product will be an integer.

Thus, $\frac{a}{b}$ multiplied by bn (126), $= \frac{abn}{b} = an$.

To perform the multiplication, *divide the multiplier by the denominator and multiply the quotient by the numerator.*

EXERCISE XXX.

1. Multiply $\frac{5}{9}$ by 18.

$$18 \div 9 = 2; 2 \times 5 = 10, \text{ Ans.}$$

2. Multiply $\frac{x-1}{x+3}$ by $x^2 + 5x + 6$.

$$(x^2 + 5x + 6) \div (x + 3) = x + 2; (x + 2) \times (x - 1) = x^2 + x - 2, \text{ Ans.}$$

Simplify:

3. $\frac{5}{7} \times 28.$

4. $\frac{x+3}{x+2} \times (x^2 - 4).$

5. $\frac{11}{15} \times 45.$

6. $\frac{5(a-b)}{(a+b)^2} \times 3(a+b)^3.$

7. $\frac{5}{a} \times (a^2 + a).$

8. $\frac{7c}{a(a-1)} \times ac(a^2 - 1).$

9. $\frac{b}{11} \times 22x.$

10. $\frac{x+2}{x-3} \times (x^2 - x - 6).$

11. $\frac{a^2b}{2c^2x} \times 6c^3x^2.$

12. $\frac{y-5}{y-3} \times (y^2 + 2y - 15).$

13. $\frac{x-a}{x+a} \times (x+a)^2.$

14. $\frac{a-b}{a+b} \times (a^3 + b^3).$

15. $\frac{3a^3b^2c}{5nx^2} \times 30a^2n^2x^2.$

16. $\frac{x^2 + y^3}{x^2 - y^3} \times (x^6 - y^9).$

Simplify:

17. $5\frac{2}{3} \times 9.$

$$5 \times 9 = 45; \quad \frac{2}{3} \times 9 = 6; \quad 45 + 6 = 51 \text{ Ans.}$$

18. $7\frac{3}{4} \times 8.$ 19. $\left(x + 1 + \frac{3x + 5}{6x}\right) \times 12x.$

20. $5\frac{4}{7} \times 14.$

21. $\left(x + 3 + \frac{5x^2 + 6x}{x + 1}\right) \times (x^2 - 1).$

22. $9\frac{3}{5} \times 20.$

23. $\left(2x - 7 + \frac{3x^2 - 7x + 5}{x + 3}\right) \times (x^2 + x - 6.)$

24. $\left(\frac{2}{3} + 5\frac{1}{2} - 2\frac{3}{4}\right) \times 12.$

Art. 107. $\left(\frac{2}{3} + 5\frac{1}{2} - 2\frac{3}{4}\right) \times 12 = 8 + 66 - 33 = 41. \text{ Ans.}$

25. $\left(3\frac{1}{2} + \frac{5}{6} - \frac{7}{12}\right) \times 12.$

26. $\left(5\frac{3}{4} - 4\frac{5}{6} + \frac{7}{8}\right) \times 24.$

27. $\left(\frac{x + 1}{3x} + \frac{3x - 7}{4x} - \frac{5}{6}\right) \times 12x.$

28. $\left(\frac{x + 3}{5} + \frac{x^2 - 7x}{x + 4} - \frac{x^2 - 5x}{x - 4}\right) \times 5(x^2 - 16).$

29. $\left(\frac{3}{5} \times \frac{7}{8}\right) \times 40.$

Art. 103, $\left(\frac{3}{5} \times \frac{7}{8}\right) \times 40 = \frac{3}{5} \times \left(\frac{7}{8} \times 40\right) = \frac{3}{5} \times 35 = 21, \text{ Ans.}$

Simplify:

$$30. \left(\frac{5}{9} \times \frac{3}{7}\right) \times 63.$$

$$31. \left(\frac{a}{b} \times \frac{m}{n}\right) \times bn.$$

$$32. \left(2\frac{1}{3} \times 3\frac{1}{4}\right) \times 12.$$

$$33. \left(\frac{3x}{x+3} \times \frac{5x}{x+4}\right) \times (x^2 + 7x + 12).$$

$$34. \left(\frac{x}{7} \times \frac{x+2}{x+4} \times \frac{x-3}{x-5}\right) \times 7(x^2 - x - 20).$$

184. *To reduce a fraction to its lowest terms.*

185. A fraction is reduced to lower terms by dividing its terms by any common factor. Hence,

Rule.—*Divide the numerator and denominator by all the common factors, or by the G. C. D. of the terms.*

EXERCISE XXXI.

Reduce to lowest terms:

$$1. \frac{5a^2b^3c}{10a^3b^2}. \quad (\text{Divide both terms by } 5a^2b^2).$$

$$2. \frac{16x^3y^4z}{20a^2y^3z^2}.$$

$$3. \frac{(a+b)^2 (a+x)^5}{(a+b)^3 (a+x)^3}.$$

$$4. \frac{35a^4b^2x}{63a^4c^2x^2}.$$

$$5. \frac{5a^4 (1+x)^3 (1-x)^3}{15a^2 (1+x)^2 (1-x)}.$$

$$6. \frac{39m^3n^5x}{65a^2m^4x}.$$

$$7. \frac{51a^2c^3 (a+x)^6 (a-x)^4}{85a^5c (a+x)^2 (a-x)^7}.$$

$$8. \frac{x^2 + 4x - 21}{x^2 - x - 56} = \frac{(x-3)(x+7)}{(x-8)(x+7)} = \frac{x-3}{x-8}.$$

Reduce to lowest terms :

- | | |
|--|---|
| 9. $\frac{x^2 - 64}{x^2 - 12x + 32}$ | 10. $\frac{x^2 + x - 72}{x^2 - x - 90}$ |
| 11. $\frac{x^2 + 12x + 27}{x^2 - 81}$ | 12. $\frac{x^2 - 49}{4x(x + 7)}$ |
| 13. $\frac{x^2 + 5x - 36}{x^2 + 2x - 24}$ | 14. $\frac{a^2 - 2a - 3}{a^2 - 10a + 21}$ |
| 15. $\frac{(x + 1)^2}{x^3 + 3x^2 - x - 3}$ | 16. $\frac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$ |

When common factors cannot be determined by inspection, the G. C. D. of the terms must be found by Art. 164, and the terms divided by it.

- | | |
|--|---|
| 17. $\frac{a^3 - 3a^2 + 4a - 2}{a^3 - a^2 - 2a + 2}$ | 18. $\frac{8x^2 - 6x - 35}{12x^2 - 23x - 77}$ |
| 19. $\frac{2x^2 - xy - 15y^2}{2x^2 - 3xy - 9y^2}$ | 20. $\frac{6a^3 - 23a^2 + 16a - 3}{6a^3 - 17a^2 + 11a - 2}$ |

186. *To reduce a fraction to an integer or mixed number.*

Rule. *If possible divide the numerator by the denominator.*

EXERCISE XXXII.

Change to integral or mixed numbers :

- | | |
|----------------------------------|--|
| 1. $\frac{x^2 + 5x + 6}{x + 3}$ | 2. $\frac{12x^2 - 4xy + 10y^2}{6x - 20y}$ |
| 3. $\frac{5x^2 + 3x + 1}{x + 4}$ | 4. $\frac{a^4 + 2a^2b^2 + b^4}{a^2 - 2ab + b^2}$ |
| 5. $\frac{7a^2 - 5a + 7}{a + 6}$ | 6. $\frac{x^2 - 11x + 24}{x^2 + x - 12}$ |

Change to integral or mixed numbers:

$$7. \frac{a^3 + b^3}{a + b}.$$

$$8. \frac{x^3 + 2x^2 - 63x}{x^2 - 2x - 35}.$$

$$9. \frac{a^4 + b^4}{a + b}.$$

$$10. \frac{12x^3 + 4x^2 - 27x - 9}{2x^2 + 5x + 3}.$$

187. To reduce an integer or fraction to a required denominator.

Rule. Multiply by the required denominator, and write the product over that denominator.

This does not change the value, as it is equivalent to multiplying and then dividing by the same number.

EXERCISE XXXIII.

Reduce:

$$1. 5 \text{ to } 4\text{ths. } 5 \times 4 = 20, \frac{20}{4}.$$

$$2. \frac{3}{5} \text{ to } 10\text{ths. } \frac{3}{5} \times 10 = 6, \frac{6}{10}.$$

$$3. \frac{a}{b} \text{ to } bn\text{ths. } \frac{a}{b} \times bn = an, \frac{an}{bn}.$$

$$4. \frac{7}{12} \text{ to } 36\text{ths. } \quad 5. \frac{a+1}{a-1} \text{ to } (a^2-1)\text{ths.}$$

$$6. \frac{a^2 b}{nx} \text{ to } n^2 x^3 \text{ths. } \quad 7. \frac{2x+1}{x+3} \text{ to } (x^2-9)\text{ths.}$$

$$8. \frac{5ac}{6bx} \text{ to } 12bx\text{ths. } \quad 9. \frac{x-5}{x+4} \text{ to } (x^2+x-12)\text{ths.}$$

$$10. \frac{x^2+5x+3}{x+6} \text{ to } (x^2+x-30)\text{ths.}$$

188. *To reduce a mixed number to an improper fraction.*

EXERCISE XXXIV.

1. Reduce $a + \frac{n}{d}$ to d ths

$$\left(a + \frac{n}{d}\right) \times d = ad + n, \frac{ad + n}{d}.$$

Rule. Multiply the integral part by the denominator, to the product add the numerator, and place the sum over the denominator.

Reduce to the fractional forms:

$$2. \quad 2x + 1 + \frac{3x + 1}{x} \qquad 3. \quad x + 2 + \frac{(x - 2)^2}{x + 2}$$

$$4. \quad 2x + 1 - \frac{3x + 1}{x} \qquad 5. \quad x + 3 - \frac{(x - 3)^2}{x + 3}$$

$$6. \quad ab + \frac{x - a^2b^2}{ab} \qquad 7. \quad x - 1 + \frac{x - 1}{x + 1}$$

$$8. \quad x^2 + 3x + 2 + \frac{x^3 - 3x^2 + 2x + 4}{x - 2}$$

$$9. \quad a^2 - 2a + 3 - \frac{a^3 - 3a^2 + 4a - 3}{a - 1}$$

189. *To reduce fractions to a common denominator, or quotients to a common divisor.*

EXERCISE XXXV.

1. Reduce $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{5}{6}$ to 12ths.

Multiply each fraction by 12, we get 8, 9, 10.

Divide each of these by 12, we get $\frac{8}{12}$, $\frac{9}{12}$, $\frac{10}{12}$.

Rule. 1^o, Find the L. C. M. of the denominators.
 2^o, Multiply each fraction by this L. C. M. 3^o,
 Write the L. C. M. under each product.

Reduce to a common denominator:

$$2. \quad \frac{3}{4}, \quad \frac{5x}{6a}, \quad \text{and} \quad \frac{7c}{8a^2}.$$

The L. C. M. of 4, 6a and 8a² is 24a².

$$\left(\frac{3}{4}, \frac{5x}{6a}, \frac{7c}{8a^2}\right) \times 24a^2 = 18a^2, 20ax, 21c;$$

$$(18a^2, 20ax, 21c) \div 24a^2 = \frac{18a^2}{24a^2}, \frac{20ax}{24a^2}, \frac{21c}{24a^2}, \text{ Ans.}$$

$$3. \quad \frac{2}{3}, \quad \frac{3a}{5} \text{ and } \frac{7x}{10}.$$

$$4. \quad \frac{2x-5}{8} \text{ and } \frac{3x+7}{12}.$$

$$5. \quad \frac{7x}{3}, \quad \frac{a}{b} \text{ and } \frac{c}{5a}.$$

$$6. \quad \frac{a+2b}{3a} \text{ and } \frac{2a-3b}{6b}$$

$$7. \quad m, \quad \frac{n}{m} \text{ and } \frac{x}{3y}.$$

$$8. \quad \frac{a+b}{a-b} \text{ and } \frac{a-b}{a+b}.$$

$$9. \quad \frac{5}{2a}, \quad \frac{7}{3a^2} \text{ and } \frac{9}{8x}.$$

$$10. \quad \frac{2x+3y}{2x-3y} \text{ and } \frac{2x-3y}{2x+3y}.$$

$$11. \quad \frac{9}{8(1+x)},$$

$$\frac{3x}{4(1+x)^2}$$

$$\frac{x^2}{2(1+x)^3}.$$

$$12. \quad \frac{x^2}{x^2-9}, \quad \frac{x+2}{x^2+3x}, \quad \frac{x-3}{x^2-3x}.$$

$$13. \quad \frac{x+3}{x^2+3x+2}, \quad \frac{x+2}{x^2+4x+3}, \quad \frac{x+1}{x^2+5x+6}.$$

ADDITION AND SUBTRACTION.

EXERCISE XXXVI.

1. Add $\frac{2}{3}$ and $\frac{5}{6}$.

The L. C. M. of 3 and 6 is 6. $\left(\frac{2}{3} + \frac{5}{6}\right)$ multiplied by 6 =
 $(4 + 5) = 9$. 9 divided by 6 = $\frac{9}{6} = 1\frac{1}{2}$, Ans.

2. Simplify $\frac{x+1}{x^2-4} + \frac{3}{2(x+2)}$.

The L. C. M. of $x^2 - 4$ and $2(x^2 - 2)$ is $2(x^2 - 4)$

Multiply $\frac{x+1}{x^2-4}$ by $2(x^2-4)$, $2x+2$

Multiply $\frac{3}{2(x+2)}$ by $2(x^2-4)$, $\frac{3x-6}{1}$

Add, $5x-4$

Divide this sum by $2(x^2-4)$, $\frac{5x-4}{2(x^2-4)}$

190. To add fractions: *Multiply each fraction by the L. C. M. of the denominators, add the products, and divide the sum by that L. C. M.*

3. From $\frac{5}{6}$ take $\frac{3}{4}$.

The L. C. M. of 6 and 4 is 12. $\left(\frac{5}{6} - \frac{3}{4}\right)$ multiplied by 12
 $= (10 - 9) = 1$. 1 divided by 12 = $\frac{1}{12}$ Ans.

4. Simplify $\frac{a+4}{3a} - \frac{b-6}{6b}$

The L. C. M. of $3a$ and $6b$ is $6ab$

Multiply $\frac{a+4}{3a}$ by $6ab$,

$$2ab + 8b.$$

Multiply $\frac{b-6}{6b}$ by $6ab$,

$$\underline{ab - 6a}$$

Subtract,

$$ab + 6a + 8b$$

Divide this by $6ab$,

$$\frac{ab + 6a + 8b}{6ab}$$

191. To subtract one fraction from another:

Multiply each fraction by the L. C. M. of the denominators, subtract the product of which the subtrahend is a factor from the other, and divide the difference by that L. C. M. Or, reverse the subtrahend and proceed as in addition.

Simplify:

5. $\frac{2x}{3} + \frac{3x}{4}$.

6. $\frac{3}{x^2 + x} - \frac{5}{x^2 - x}$.

7. $\frac{7x}{8} - \frac{5x}{6}$.

8. $\frac{3}{x+3} + \frac{1}{x^2-9}$.

9. $\frac{x+1}{5} + \frac{x-1}{10}$.

10. $\frac{x+1}{x^2-16} - \frac{2}{x+4}$.

11. $\frac{x+3}{4} - \frac{2x-5}{7}$.

12. $\frac{a}{a+b} + \frac{a}{a-b}$.

13. $\frac{5x+3}{12} + \frac{3x-2}{8}$.

14. $\frac{a}{a+b} - \frac{a}{a-b}$.

15. $\frac{3x}{4a} - \frac{2x-3}{4b}$.

16. $\frac{a+b}{a-b} + \frac{a-b}{a+b}$.

17. $\frac{5a-b}{2b} + \frac{7a+3b}{6b}$, 18. $\frac{a+b}{a-b} - \frac{a-b}{a+b}$.

Simplify:

$$19. \frac{6}{3x-1} + \frac{5}{3x+1} - \frac{15x+1}{9x^2-1}.$$

$$20. \frac{4}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}.$$

$$21. \frac{3x-2y}{5x} + \frac{5x-7y}{10x} + \frac{8x+2y}{25}.$$

$$22. \frac{4x+5}{3} - \frac{3x-7}{5x} + \frac{9}{12x^2}.$$

$$23. \frac{x+2}{(x+3)(x+4)} + \frac{x-3}{(x+4)(x+5)}.$$

$$24. \frac{x-6}{(x+4)(x-5)} - \frac{x+7}{(x-5)(x+6)}.$$

$$25. \frac{x+3}{(x+1)(x+2)} - \frac{x+1}{(x+2)(x+3)} + \frac{x+2}{(x+1)(x+3)}.$$

$$26. \frac{x+4}{x^2+5x+6} + \frac{x+3}{x^2+6x+8} - \frac{x+2}{x^2+7x+12}.$$

$$27. \frac{x+6}{x^2-9x+20} - \frac{2(x+5)}{x^2-10x+24} + \frac{x+4}{x^2-11x+30}.$$

$$28. \frac{2}{x} - \frac{3}{2x-1} - \frac{2x-3}{4x^2-1}.$$

$$29. \frac{1}{1-x} - \frac{1}{x+2} - \frac{3}{(x+2)^2}.$$

$$30. \frac{3+2x}{2-x} - \frac{2-3x}{2+x} + \frac{16x-x^2}{x^2-4}.$$

$$31. \frac{3}{1-2x} - \frac{7}{1+2x} - \frac{4-20x}{4x^2-1}.$$

Simplify:

$$32. \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}.$$

$$33. \frac{(b-a)}{x-b} - \frac{a-2b}{x+b} + \frac{3x(a-b)}{x^2-b^2}.$$

$$34. \frac{3a-4b}{7} - \frac{2a-b-c}{3} + \frac{15a-4c}{12} - \frac{a-4b}{21}.$$

$$35. \frac{1}{a+b} + \frac{b}{a^2-b^2} - \frac{a}{a^2+b^2}.$$

$$36. \frac{a}{a-x} + \frac{3a}{a+x} - \frac{2ax}{a^2-x^2}.$$

$$37. \frac{a+b}{(b-c)(c-a)} + \frac{b+c}{(c-a)(a-b)} + \frac{c+a}{(a-b)(b-c)}.$$

$$38. \frac{bc}{(c-a)(a-b)} + \frac{ca}{(a-b)(b-c)} + \frac{ab}{(b-c)(c-a)}.$$

$$39. \frac{1}{(b-c)(c-a)} + \frac{1}{(c-a)(a-b)} + \frac{1}{(a-b)(b-c)}.$$

$$40. \frac{a^2-bc}{(a+b)(a+c)} + \frac{b^2-ac}{(b+c)(b+a)} + \frac{c^2-ab}{(c+a)(c+b)}.$$

$$41. \frac{a^2-bc}{(a-b)(a-c)} + \frac{b^2+ac}{(b+c)(b-a)} + \frac{c^2+ab}{(c-a)(c+b)}.$$

MULTIPLICATION.

EXERCISE XXXVII.

1. Multiply $\frac{3}{5}$ by $\frac{7}{4}$.

$$\left(\frac{3}{5} \times \frac{7}{4}\right) \times (5 \times 4), \text{ Art. 188, Exs. 29 to 34, } = 3 \times 7,$$

$$(3 \times 7) \div (5 \times 4) = \frac{3 \times 7}{5 \times 4}, \text{ Ans.}$$

2. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

$$\left(\frac{a}{b} \times \frac{c}{d}\right) \times (b \times d) = a \times c;$$

$$(a \times c) \div (b \times d) = \frac{a \times c}{b \times d}, \text{ Ans.}$$

192. Rule. *Cancel common factors in the numerators and denominators, then multiply numerators together for a new numerator, and denominators together for a new denominator.*

Mixed numbers must be reduced to improper fractions.

Simplify:

3. $\frac{6}{35} \times \frac{55a}{24b}$

4. $2\frac{1}{7} \times 2\frac{1}{3}$

5. $\frac{9}{8} \times \frac{4(x+a)}{15(x-a)}$

6. $\left(a + \frac{x^2}{a}\right) \times \frac{3a}{4}$

7. $\frac{25x}{42a} \times \frac{49a}{30x}$

8. $\left(a - \frac{x^2}{a}\right) \times \frac{a}{a-x}$

9. $\frac{2(a+x)}{3(a-x)} \times \frac{6(a-x)}{8(a+x)}$

10. $\frac{3-x}{2-y} \times \frac{3(y-2)}{2(x-3)}$

11. $\frac{3x^2}{4a^3} \times \frac{8a^5}{9x^4}$

12. $\frac{3x}{4y} \times \frac{8a}{7b} \times \frac{21b^2y}{4a^2x^2}$

Simplify:

$$13. \frac{5(a+x)^4}{7(x+y)^5} \times \frac{14(x+y)^2}{15(a+x)}.$$

$$14. \frac{x^2-9}{x+5} \times \frac{x^2-25}{x-3}.$$

When the terms are polynomials, it is best to resolve them into their factors, in order to detect common factors in the numerators and denominators.

$$15. \frac{x^2+5x+6}{x^2+7x+10} \times \frac{x^2+9x+20}{x^2-3x-28}.$$

$$\frac{(x+2)(x+3)(x+4)(x+5)}{(x+2)(x+5)(x+4)(x-7)} = \frac{x+3}{x-7}.$$

$$16. \frac{x^2-1}{x^2-4} \times \frac{x^2-x-6}{x^2+2x+1}.$$

$$17. \frac{x^2-4x+3}{x^2+4x+3} \times \frac{x^2+5x+6}{x^2-7x+12}.$$

$$18. \frac{x^2-9}{x^2-16} \times \frac{(x+4)^2}{(x-3)^2} \times \frac{x^2-7x+12}{x^2+7x+12}.$$

$$19. \frac{x^2+8x+15}{x^2+5x-14} \times \frac{x^2-7x+10}{x^2-x-12} \times \frac{x^2+3x-28}{x^2-25}.$$

$$20. \frac{3ax}{4by} \times \frac{a^2-x^2}{c^2-x^2} \times \frac{bc+bx}{a^2+ax} \times \frac{c-x}{a-x}.$$

$$21. \frac{1-x^2}{1+y} \times \frac{1-y^2}{x+x^2} \times \left(1 + \frac{x}{1-x}\right).$$

$$22. \frac{a^4-b^4}{(a-b)^2} \times \frac{a-b}{a^2+ab}.$$

$$23. \left(\frac{x^2}{a^2} - \frac{x}{a} + 1\right) \times \left(\frac{x^2}{a^2} + \frac{x}{a} + 1\right).$$

$$24. (x^2-x+1) \times \left(1 + \frac{1}{x} + \frac{1}{x^2}\right).$$

$$25. \left(\frac{a^3-b^3}{a^3+b^3}\right) \left(\frac{a+l}{a-b}\right) \left(\frac{a^2-ab+b^2}{a^2+ab+b^2}\right).$$

Simplify:

$$26. \frac{6x^2 - x - 1}{5x^2 + 23x + 12} \times \frac{10x^2 + 9x + 2}{3x^2 - 11x - 4}.$$

$$27. \frac{x^2 - 2x - 15}{x^2 - 2x - 63} \times \frac{x^2 + 6x - 7}{x^2 + 3x - 40} \times \frac{x^2 - x - 72}{x^2 + 4x + 3}.$$

DIVISION.

EXERCISE XXXVIII.

1. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

$\frac{a}{b}$ multiplied by $bd = ad$; $\frac{c}{d}$ multiplied by $bd = bc$.

Hence (126), $\frac{a}{b} \div \frac{c}{d} = ad \div bc = \frac{ad}{bc}$, Ans.

193. Rule I.—*Multiply both dividend and divisor by the L. C. M. of their denominators, and divide the former product by the latter; or,*

II.—*Invert the divisor and proceed as in multiplication.*

Divide:

$$2. \frac{3x}{4} \text{ by } \frac{5x}{6}.$$

$$3. \frac{3p}{2p-2} \text{ by } \frac{2p}{p-1}.$$

$$4. \frac{4x-2}{3} \text{ by } \frac{2x}{5}.$$

$$5. \frac{a-b}{a^2+ab} \text{ by } \frac{a^2-ab}{a^2-b^2}.$$

$$6. \frac{x^2-1}{5} \text{ by } \frac{x+1}{10}.$$

$$7. \frac{a^3-x^3}{a^3+x^3} \text{ by } \frac{(a-x)^2}{(a+x)^3}.$$

$$8. \frac{x^2-9}{x^2-16} \text{ by } \frac{x-3}{x+4}.$$

$$9. \frac{m^2-n^2}{c^3+8} \text{ by } \frac{n-m}{c+2}.$$

$$10. \frac{x^3-x}{4-x^2} \text{ by } \frac{x-1}{x-2}.$$

Divide:

$$11. \quad \frac{x^2 + x - 2}{x^2 - 7x} \text{ by } \frac{x^2 + 2x}{x^2 - 13x + 42}.$$

Simplify:

$$12. \quad \left(\frac{a^2}{b^2} - \frac{b^2}{a^2} \right) \div \left(\frac{a}{b} + \frac{b}{a} \right).$$

$$13. \quad \frac{x^2 + xy + y^2}{x^2 - xy + y^2} \div \frac{x + y}{x - y}.$$

$$14. \quad \frac{2a}{a+b} \div \left(1 - \frac{a-b}{a+b} \right).$$

$$15. \quad \frac{(x^2 - 8)^2}{(x^2 + 8)^2} \div \frac{x^4 + 2x^2 - 80}{x^4 - 2x^2 - 80}.$$

$$16. \quad \left(x^2 - \frac{1}{x^2} \right) \div \left(x + \frac{1}{x} \right).$$

$$17. \quad \frac{x^2 - 14x + 49}{x^2 - 12x + 36} \div \frac{x^2 - 15x + 56}{x^2 - 11x + 30}.$$

$$18. \quad \left(x^3 - \frac{1}{x^3} \right) \div \left(x - \frac{1}{x} \right).$$

$$19. \quad \left(x^2 + 2 + \frac{1}{x^2} \right) \div \left(x + \frac{1}{x} \right).$$

$$20. \quad \left(x^2 + 1 + \frac{1}{x^2} \right) \div \left(x - 1 + \frac{1}{x} \right)$$

$$21. \quad \left(\frac{x^2}{2a^2} - 4 + \frac{6a^2}{x^2} \right) \div \left(\frac{x}{2a} - \frac{3a}{x} \right)$$

$$22. \quad (a^2 - b^2 - c^2 + 2bc) \div \frac{a + b - c}{a + b + c}.$$

$$23. \quad (a^2 - b^2 - c^2 - 2bc) \div \frac{a + b + c}{a + b - c}.$$

$$24. \quad \left(x^2 - 3ax - 2a^2 + \frac{12a^3}{x + 3a} \right) \div \left(3x - 6a - \frac{2x^2}{x + 3a} \right).$$

COMPLEX FRACTIONS.

EXERCISE XXXIX.

194. 1. Simplify $\frac{\frac{2}{3}}{\frac{3}{4}}$.

$$\frac{\frac{2}{3}}{\frac{3}{4}} = \frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}.$$

2. Simplify $\frac{3 + 1\frac{1}{2}}{4\frac{1}{3} - 2\frac{1}{4}}$.

$$\frac{3 + 1\frac{1}{2}}{4\frac{1}{3} - 2\frac{1}{4}} = (3 + 1\frac{1}{2}) \div (4\frac{1}{3} - 2\frac{1}{4}) = \frac{9}{2} \div \frac{25}{12} = \frac{9}{2} \times \frac{12}{25} = \frac{54}{25}.$$

It is often shorter to multiply both terms of the fraction by the L. C. M. of the denominators of the fractions contained in the terms.

Thus, in (1) multiply both terms by 12, and in (2) multiply both terms by 12.

Simplify:

3. $\frac{\frac{3}{4} + \frac{3}{5}}{\frac{2}{3} - \frac{1}{2}}$.

4. $\frac{x - 1 + \frac{1}{x + 1}}{x + 1 + \frac{1}{x - 1}}$.

5. $\frac{\frac{2x}{5} + \frac{3x}{8}}{3x + \frac{7x}{10}}$.

6. $\frac{3x + \frac{27x}{2x + 3}}{2x + \frac{18x}{2x - 3}}$.

Simplify:

$$7. \frac{\frac{a+x}{a-x} + \frac{a-x}{a+x}}{\frac{a+x}{a-x} - \frac{a-x}{a+x}}$$

$$8. \frac{\frac{x+1}{x-1} + \frac{x-1}{x+1}}{\frac{x+1}{x-1} - \frac{x-1}{x+1}}$$

$$9. \frac{\frac{x}{x+y} + \frac{y}{x-y}}{\frac{x}{x-y} - \frac{y}{x+y}}$$

$$10. \frac{\frac{x^2 - 17x + 72}{x^2 + 22x + 120}}{\frac{x^2 - 21x + 108}{x^2 + 18x + 80}}$$

$$11. \frac{\frac{a+b}{a-b} + \frac{a^2+b^2}{a^2-b^2}}{\frac{a-b}{a+b} - \frac{a^3-b^3}{a^3+b^3}}$$

$$12. \frac{\frac{c-b}{c+b} - \frac{c^3-b^3}{c^3+b^3}}{\frac{c+b}{c-b} + \frac{c^2+b^2}{c^2-b^2}}$$

$$13. \frac{\frac{x^2+y^2}{x^2-y^2} - \frac{x^2-y^2}{x^2+y^2}}{\frac{x+y}{x-y} - \frac{x-y}{x+y}}$$

$$14. \frac{\frac{a+b}{a-b} + \frac{a-b}{a+b}}{\frac{a^2+b^2}{a^2-b^2} - \frac{a^2-b^2}{a^2+b^2}}$$

$$15. \frac{1}{x + \frac{1}{x + \frac{1}{x}}}$$

$$16. \frac{1}{a + \frac{1}{b + \frac{1}{c}}}$$

$$17. \frac{a}{b + \frac{c}{d + \frac{e}{f}}}$$

$$18. \frac{1}{x + \frac{1}{1 + \frac{x+1}{3-x}}}$$

CHAPTER V.

SIMPLE EQUATIONS.

DEFINITIONS AND PRINCIPLES.

195. An **Equation** is an expression of the equality of two quantities.

Thus, $12 - 7 = 5$,
is an equation, as it expresses the equality of the compound number $12 - 7$ and the simple number 5.

196. The **First Member** or **Side** of an equation is the quantity on the left of the sign $=$; and the **Second Member** or **Side** is the quantity on the right of the sign $=$.

197. To **Verify** an equation is to reduce both sides to their simplest forms, and thus show that they are equal.

EXERCISE XI

Verify:

1. $6 + 5 - 2 = 8 + 1$. 2. $10 - 7 + 3 = 11 - 5$.

3. $14 - (5 + 3) = 4 - (-2)$.

4. $10 - (11 - 3) = -(3 - 5)$.

5. $(6 - 2)(8 - 5) = -3(-4)$.

6. $18 - (11 - 2)(10 - 8) = 0$.

7. $\frac{7-1}{2} - \frac{7-3}{4} = \frac{7-3}{2}$.

Verify:

$$8. \frac{1}{4-2} + \frac{1}{5-2} = \frac{9-4}{3(4-2)}.$$

$$9. \frac{a-1}{3} + \frac{a-1}{2} = \frac{5(a-1)}{6}.$$

$$10. \frac{3x-4}{4} - \frac{2(x+1)}{3} = \frac{x-20}{12}.$$

Let the student ascertain by trial what number put in the place of, or substituted for x , will satisfy the following:

$$11. 9 - x = 5. \quad 12. 5 \times 4 - 3x = 14.$$

$$13. 3x - 7 = 2x + 3. \quad 14. 5x - (2x - 1) = 10.$$

$$15. \frac{x}{2} + \frac{x}{3} = 10. \quad 16. \frac{x-1}{2} - \frac{x-3}{4} = \frac{x-1}{3}.$$

198. Known Numbers (5) are usually represented by figures, or by the first letters of the alphabet, as $a, b, c, \&c.$

199. Unknown Numbers (6) are usually represented by the last letters of the alphabet, as $x, y, z.$

200. An Identical Equation is one in which the two members are equal, whatever numbers the letters stand for.

$$\text{As } (x + a)^2 = x^2 + 2ax + a^2.$$

201. A Conditional Equation is one which is true only on condition that the letters representing unknown numbers stand for particular values.

As, ex's 11 to 16, Art. 197.

202. The **Value** of the unknown number (or a Root of the equation) is the number which substituted for it will satisfy the equation.

203. A **Simple Equation**, or an equation of the *first degree*, is one which contains only the first power of the unknown number.

204. To **Solve** an equation is to find its root, or the value of the unknown number.

205. The third particular in which Algebra is an extension of Arithmetic (**11, 69**) is the solution of problems by means of equations and their transformations.

206. To **Transform** an equation is to change its form without destroying the equality.

207. The transformations of equations are based on the following:

Principle. If equal changes be made in both members of an equation, the results will be equal.

EXERCISE XLI.

First Form: $ax = b$.

1. Find the value of x in $5x = 20$.

Dividing both members by 5 $5 \mid 5x = 20$
 is making equal changes in $x = 4$.
 both members. Hence,

208. Rule. *Divide both members by the coefficient of x .*

Find the value of x in:

2. $6x = 18.$ 6. $ax = b.$ 10. $(a - b)x = d.$
 3. $7x = 35.$ 7. $mx = c.$ 11. $(a + n)x = m.$
 4. $12x = 27.$ 8. $rx = ar.$ 12. $(a + 5)x = a^2 - 25.$
 5. $9x = 42.$ 9. $m^2x = cm^3.$ 13. $(n - 3)x = n^3 - 27.$

Second Form: $ax + bx - cx = m - n.$

14. Solve $7x - 3x + 5x = 30 - 5 + 11.$

Reducing by uniting similar terms, we have

$$9x = 36, \therefore x = 4.$$

209. Rule. *Reduce each member to a simple number, and proceed as with the First Form.*

15. $9x + 11x - 5x = 20 - 3 + 13.$
 16. $11x - 5x - 2x = 13 + 7 + 12.$
 17. $6x - 8x + 7x - 2x = 5 + 19 - 6 + 9.$
 18. $5x + 4x + 3x - 2x = 17 - 7 + 28 - 8 + 5.$
 19. $6x - (3x - 7x + 8x) = 3 + 7 - (14 - 6 + 8).$
 20. $ax - bx + cx = m - n.$
 $(a - b + c)x = m - n. \therefore x = \frac{m - n}{a - b + c}.$
 21. $ax + bx - nx = m + r.$
 22. $mx - nx = n + m - a.$
 23. $ax + bx = a^2 - b^2.$ 24. $mx - nx = m^2 - n^2.$
 25. $bx + cx = b^3 + c^3.$ 26. $cx + x = c^3 + 1.$
 27. $ax + bx = m(a + b) + n(a + b).$
 28. $ax - cx = c(a^2 - c^2) + a(a - c)^2.$
 29. $a^2x + ax + x = a^4 + a^2 + 1.$

Third Form: $ax + m - cx = a - bx + n$.

210. To **Transpose** a term is to carry it from one side to the other, which is done by subtracting it from both members (207).

Thus, take the equation $x + 4 = 8$.
 Subtract $+ 4$ from both members, $x = 8 - 4$.
 Again, take the equation $x - 4 = 8$.
 Subtract $- 4$ from both members, $x = 8 + 4$.

Hence, a term may be transposed by changing its sign and writing it in the other member.

30. Solve $5x - 7x + 12 = -6x + 28$.

Transpose, $5x - 7x + 6x = -12 + 28$.
 Reduce, $4x = 16$.
 Divide by 4, $x = 4$.

211. Rule.—*Transpose all the terms containing the unknown number to the first member, and all the others to the second member, and then proceed as with the Second Form.*

Find the values of x in:

- 31.** $7x - 18 + 5x = + 18$. **35.** $cx - b + n - mx = 0$.
32. $8x + 19 - 5 = x$. **36.** $cx + n^3 = nx + c^3$.
33. $11x - 3x - 37 = 5$. **37.** $cx - n^3 = c^3 - nx$.
34. $ax + m = n - bx$. **38.** $5x - 20 = 50 - 2x$.
39. $5x + 15 - 4x + 12 = 10x - 50 - 40$.
40. $3x + 18 - (16 - 3x) = 50$.
41. $6x + 6 + (4x + 8) = 192 - (15x + 3)$.
42. $3(3x - 2) + 5(x - 6) = 18x - 4$.
43. $6(x + 1) + 4(x + 2) - 3(5 - x) = 12 \times 14$.
44. $p(x - p) = q(x - q)$.
45. $(x + a)(x - a) = (x - a)(x - b) + ab + b^2$.

Fourth Form: $\frac{x}{n} + \frac{a}{m} = \frac{b}{r} - \frac{ax}{m}.$

46. Solve $\frac{7x + 5}{6} - \frac{5x - 6}{4} = \frac{8 - 5x}{12}.$

Multiply both members by 12, the L. C. M. of the denominators, and we have:

$$14x + 10 - 15x + 18 = 8 - 5x.$$

Transpose, $14x - 15x + 5x = 8 - 18 - 10$

Reduce, $4x = -20$

Divide by 4 $x = -5.$

212. Rule.—*Multiply each term by the L. C. M. of the denominators, and proceed as with the Third Form.*

If a fraction is preceded by the sign —, the sign of every term of the numerator must be changed when the denominator is removed.

The sign of all the terms of both members may be changed, as it is equivalent to multiplying both members by — 1.

47. $\frac{5x}{6} = 4 - \frac{x}{2}.$

48. $\frac{x}{2} + \frac{x}{3} = 9.$

49. $\frac{3}{2} - \frac{x}{5} = \frac{3x}{5} - \frac{5}{2}.$

50. $\frac{x}{2} - \frac{x}{4} + 15 = \frac{x}{8} - \frac{x}{6} + 22.$

51. $\frac{x+3}{4} + 2 = \frac{x-3}{5} + \frac{x-5}{2}.$

52. $\frac{x-3}{4} - 6 - \frac{x-1}{5} = \frac{x-5}{3} - 8.$

53. $\frac{x+1}{2} + \frac{x+2}{3} = 16 - \frac{5x+1}{4}.$

$$54. \frac{4x}{5} - \frac{5x}{4} + 18 = \frac{1}{9} (4x + 1).$$

$$55. \frac{x+1}{2} - \frac{5-x}{4} = 14 - \frac{x+2}{3}.$$

$$56. \frac{5x-4}{6} - \frac{7x+6}{12} = \frac{2x-1}{3} - \frac{3x-2}{4}.$$

$$57. \frac{1}{8} (28 - 9x) - \frac{1}{4} (3x - 11) = 14\frac{3}{4} - 4x.$$

$$58. \frac{1}{7} (5x - 1) - \frac{1}{5} (9x - 7) - \frac{1}{11} (5 - 9x) = 0.$$

$$59. \frac{1}{4} x - \frac{1}{3} (2x - 9) - \frac{1}{6} (5x + 8) = 0.$$

$$60. \frac{3x-7}{5} + \frac{25-4x}{9} = \frac{5x-14}{3}.$$

$$61. 4x + 17\frac{1}{2} - \frac{1}{2} (7x + 2) = 19x.$$

$$62. \frac{x}{7} - \frac{x-5}{11} + 5 = x - \left(\frac{2x}{77} + 1 \right).$$

$$63. \frac{4x+12}{x+2} + 7 = \frac{37}{x+2}.$$

$$64. \frac{6x+7}{15} = \frac{2x-2}{7x-6} + \frac{2x+1}{5}.$$

$$65. \frac{2}{2x-5} - \frac{6}{3x-1} = -\frac{1}{x-3}.$$

$$66. \frac{6x+1}{15} - \frac{2x-4}{7x-16} = \frac{2x-1}{5}.$$

$$67. \left(x + \frac{5}{2} \right) \left(x - \frac{3}{2} \right) + \frac{3}{4} = (x+5)(x-3).$$

$$68. \left(x - \frac{5}{2} \right) \left(x + \frac{3}{2} \right) - \frac{93}{4} = (x-5)(x+3).$$

$$69. \frac{1}{x-2} - \frac{1}{x-4} = \frac{1}{x-6} - \frac{1}{x-8}.$$

$$70. \frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-5}{x-6} - \frac{x-6}{x-7}.$$

$$71. \frac{x}{a} + \frac{b-x}{x} = \frac{x^2-c}{ax}.$$

$$72. \frac{x+a}{x+c} = \frac{x+c}{x+a}.$$

$$73. ax - \frac{3a-bx}{3} = \frac{1}{2}.$$

$$74. 6a - \frac{ax-b}{3} = x.$$

$$75. \frac{x}{a+1} - b = \frac{x}{a-1}.$$

$$76. \frac{c}{a^2-b^2} = \frac{x}{a-b} - \frac{2+x}{a+b}.$$

$$77. (a+x)(b+x) - a(b+c) = \frac{a^2c}{b} + x^2.$$

$$78. \frac{x+a}{x+b} - \frac{x+b}{x+a} = \frac{2(a+b)(a-b)}{(x+b)(x+a)}.$$

$$79. \frac{25-\frac{1}{3}x}{x+1} + \frac{16x+\frac{4}{5}}{3x+2} = \frac{23}{x+1} + 5.$$

$$80. \frac{3abc}{a+b} + \frac{a^2b^2}{(a+b)^3} + \frac{(2a+b)b^2x}{a(a+b)^2} = 3cx + \frac{bx}{a}.$$

PROBLEMS.

213. A Problem is a question proposed for solution.

214. A Condition is a relation existing between the known and unknown quantities of a problem.

In a problem the conditions are expressed in common language.

215. The solution of a problem by Algebra consists of two parts:

1^o, The **Statement**: that is, expressing the conditions by means of equations.

2^o, The **Solution** of the equations.

No general rule can be given for the statement of a problem, but by adopting the following method the student will soon acquire the ability to make statements intelligently and readily:

To the student. Take or assume some number, and by trial ascertain whether or not it is the answer. If you can do this, it shows that you understand the conditions.

If you can not do this, ask the teacher to show you how to ascertain whether or not the assumed number is the answer: this will help you to understand the conditions.

Now take x as the representative of the unknown number, and proceed with it just as you did with the assumed number, and where it ought to be equal to so and so, make it equal to it, and you have the statement.

EXERCISE XLII.

1. Two boys have 24 marbles and the second boy has twice as many as the first; how many has each?

Trial Solution.

Suppose the first boy has	6 marbles
Then the second boy has 2 times 6, or 12	“
and both boys have	18 “
But both boys have	24 “

∴ 18 ought to be equal to 24.

But, as this is not the case, 6 is not the correct answer.

Solution.

Suppose the first boy has x marbles.

Then the second boy has 2 times x , or $2x$ “

and both boys have $x + 2x$, or $3x$ “

But both boys have 24 “

$$\therefore 3x = 24.$$

Whence $x = 8$, the number the first has,

and $2x = 16$, the number the second has.

Comparing the two solutions, we see that “ x ” takes the place of “6,” the supposed number, and “ $=$ ” the place of “ought to be equal to.”

2. James paid \$12 for a hat and coat, and the coat cost 3 times as much as the hat; what did each cost?

3. A farmer paid \$17.50 for a hog and cow, paying 6 times as much for the cow as the hog; what did each cost?

4. A man and his son earned \$6.40; what did each earn if the man earned 4 times as much as the son.

5. General Problem. The sum of two numbers is s , and the greater is t times the less; what are the numbers?

$$\text{Ans. Less} = \frac{s}{1+t}; \text{ greater} = \frac{ts}{1+t}.$$

216. A **General Problem** is one in which the given numbers are expressed by letters. The answer to a general problem is a *formula*, by which all similar questions may be solved.

Thus, to solve problem 2, we substitute 12 for s and 3 for t in the preceding formulas and obtain

$$\text{Less} = \frac{12}{1+3} = 3, \text{ greater} = \frac{3 \times 12}{1+3} = 9.$$

Problems 3 and 4 may be solved similarly.

6. William and Charles have 37 nuts and Charles has 11 more than William; how many has each.

Let $x =$ the number William has.

Then $x + 11 =$ the number Charles has,

and $x + x + 11 =$ the number both have.

But $37 =$ the number both have.

$$\therefore x + x + 11 = 37.$$

Reducing, etc., $2x = 26.$

Whence, $x = 13$, William's number,

and $13 + 11 = 24$, Charles' number.

7. A horse and cow cost \$153, and the horse cost \$75 more than the cow; what did each cost?

8. James and John have \$12, and John has \$3.70 less than James; how much has each?

9. Two boys were 173 feet apart, and traveled towards each other until they met; how far did each go if one of them went 21 feet further than the other?

10. The sum of two numbers is s and their difference is d ; what are the numbers?

11. Two boys, A and B, have 75 apples, and B has 7 more than 3 times as many as A; how many has each?

Let $x =$ the number A has.

Then $3x + 7 =$ " " B "

and $x + 3x + 7 =$ " " both have.

But $75 =$ " " " "

$$\therefore x + 3x + 7 = 75.$$

Transposing, etc. $4x = 68.$

Whence, $x = 17$, A's number,

and $75 - 17 = 58$, B's "

12. A father wishes to divide 147 peaches between two sons so that one of them shall have 13 less than 7 times as many as the other. How many must he give to each?

13. Henry lacks 27 years of being 3 times as old as Moses, and the sum of their ages is 33 years. What are their ages?

14. C and D together paid a debt of \$279 thus: D first paid \$24, and then paid 4 times as much of the remainder as C. What did each pay?

15. The sum of two numbers is s , and one of them is d more than t times the other; what are the numbers?

16. The sum of the third and fourth parts of a number is 35; what is the number?

Let x = the number.

Then $\frac{x}{3}$ = " third of the number

and $\frac{x}{4}$ = " fourth " "

and $\frac{x}{3} + \frac{x}{4}$ = " sum of the two parts.

But 35 = " " " " " "

$$\therefore \frac{x}{3} + \frac{x}{4} = 35$$

Multiplying both sides by 12,

$$4x + 3x = 420$$

Reducing, $7x = 420$

Whence, $x = 60$, the number.

17. Find the number whose fourth is 18 more than its tenth?

18. Find the number whose ninth is 15 less than its sixth.

19. Find the number whose ninth and sixth together make 15.

20. The third of a number exceeds its tenth by 28; what is the number?

21. Find the number whose m th part is d more than its n th part.

22. Find the number whose m th and n th parts are together equal to s .

23. Gordon and John have together \$60, and one-seventh of Gordon's money is equal to one-fifth of John's; how much has each?

Let x = the am't Gordon has.

Then $60 - x$ = the am't John has;

and $\frac{x}{7}$ = one-seventh of the am't Gordon has,

and $\frac{60 - x}{5}$ = one-fifth the am't John has.

$$\frac{x}{7} = \frac{60 - x}{5}$$

Whence, $x = 35$, the am't Gordon has,

and $60 - 35 = 25$, the am't John has.

24. Divide 100 into two parts such that if one part be divided by 8, and the other by 12, the quotients shall be equal?

25. What time of day is it if one-fourth of the time past noon is equal to one-fifth of the time till midnight?

26. Divide 46 into two such parts that if one part be divided by 7, and the other by 3, the sum of the parts shall be 10.

27. A house and garden cost \$850, and five times the price of the house was equal to twelve times the price of the garden. What is the price of each?

28. A lot of 15 hogs cost \$118; if the larger hogs cost \$10 each and the smaller \$6 each, how many of each kind were there?

29. The sum of two numbers is s , and the m th part of the first is equal to the n th part of the second. What are the numbers?

30. Divide s into two parts such that if one part be divided by m , and the other by n , the sum of the quotients shall be a .

31. Divide 25 into two parts such that the second divided by the first gives 2 as a quotient and 4 as a remainder.

Let x = the first part.

Then $25 - x$ = the second part.

Now, according to the condition, if $25 - x$ were less by 4, it would contain x , 2 times

$$\text{Hence, } \frac{25 - x - 4}{x} = 2.$$

Whence, $x = 7$, and $25 - x = 18$.

32. Divide 60 into two parts such that the second divided by the first part gives 5 as a quotient and 6 as a remainder.

33. Divide 20 into two parts such that three times the second divided by two times the first gives 1 as a quotient and 11 as a remainder.

34. Divide s into two parts such that the larger divided by the smaller gives q as a quotient and r as a remainder.

35. A certain sum is to be divided among A, B and C. A is to have £30 less than the half, B is to have £10 less than the third, and C is to have £8 more than the fourth. What does each receive?

36. Divide 108 into three parts such that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second and $\frac{1}{4}$ of the third shall be equal to each other.

37. Divide 180 into four parts such that the first increased by 5, the second decreased by 5, the third

multiplied by 5 and the fourth divided by 5, shall be equal to each other.

38. A and B play at a game and commence with equal sums. After A has won \$20, one-fifth of the amount he has is equal to one-third of what B has; how much had each at first?

39. A man bought a buggy at a certain price and paid twice as much for a horse; if the buggy had cost \$10 more and the horse \$55 less, then the cost of the horse would have been one-fourth more than that of the buggy; what did each cost?

40. A lady having been asked her age replied: "Three-fifths of my age lacks 29 years of being 1 year more than twice my age." What was her age?

NOTE I. $x, x + 1, x + 2$, etc., are consecutive numbers.

41. The difference of the squares of two consecutive numbers is 37; what are the numbers?

42. Find three consecutive numbers such that 10 times the first increased by 6 times the second shall be equal to 15 times the third.

43. In a basket is a number of apples. Henry takes out 10 apples and $\frac{1}{3}$ of what remains; then John takes out 20 apples and $\frac{1}{2}$ of what remains; and then they find that there are 20 apples remaining; how many were in the basket?

NOTE II.—If	$x =$ a person's age now,
then	$x - n =$ his age n years since,
and	$x + n =$ his age n years hence.

44. In 9 years a boy will be 4 times as old as he was 6 years ago. How old is he?

Let x = the number of years in his age.
 Then $x - 6$ = the number of years of his age 6 yrs. ago,
 and $x + 9$ = the number of years of his age 9 years
 hence. $\therefore x + 9 = 4(x - 6)$.
 Whence, $x = 11$.

45. A man is 3 times as old as his son, and 5 years hence he will be 5 times as old as the son was 5 years since. What is the age of each?

46. A is twice as old as B, and 5 years ago he was $2\frac{1}{2}$ times as old; what is the age of each?

47. Eleven years ago the age of A was 44 years, and of B 13 years; in how many years will A be twice as old as B?

48. A is 54 years old; B's age is $\frac{7}{9}$ of A's, and C's age is $\frac{6}{7}$ of B's; how long is it since A's age was equal to the sum of B's and C's?

49. In m years a man will be n times as old as he was r years ago. How old is he?

NOTE III.—If a man can do a work in n days,

(1) In one day he can do $\frac{1}{n}$ part of the work,

(2) In two days he can do $\frac{2}{n}$ part of the work,

(3) In x days he can do $\frac{x}{n}$ part of the work,

(4) In n days he can do $\frac{n}{n}$ ($= 1$) the whole work.

50. A can do a piece of work in 4 days, and B can do it in 6 days. How long will it take A and B together to do the work?

Let x = the number of days it will take A and B together.

Now, $\frac{1}{4}$ = the part A can do in one day.

Hence, $\frac{x}{4} =$ the part A can do in x days.

Again, $\frac{1}{6} =$ the part B can do in one day.

Hence, $\frac{x}{6} =$ the part B can do in x days.

Now as A and B do the whole work in x days, we have

$$\frac{x}{4} + \frac{x}{6} = 1.$$

Whence, $x = 2\frac{2}{3}.$

51. A can do a work in 8 days, B in 10 days, and C in 20 days; in what time will they do it, all working together?

52. A can do a work in $6\frac{2}{3}$ days, B in $7\frac{1}{2}$ days and C in $11\frac{1}{4}$ days; in what time will they do it, all working together?

53. A can do a work in 10 days, but with the help of B he can do it in 6 days; how long would it take B to do it alone?

54. A can dig a trench in one-half of the time that B can; B can dig it in two-thirds of the time that C can; all together they can dig it in 6 days; find the time it would take each of them alone?

55. Eight boxes or 24 barrels will fill a certain room; what equal number of boxes and barrels will fill the same room?

56. A box will hold 20 oranges; after putting 7 oranges and 2 peaches in the box, I observe that it is $\frac{5}{12}$ full. How many peaches will the box hold?

57. A cistern can be filled by one pipe in 10 hours, by another in 15 hours and emptied by another in 12 hours; supposing it at first empty, in what time will it be filled when all the pipes are running?

58. Two persons, A and B, could finish a work in 12 days; they worked together 7 days when A was called off and B finished it in 8 days. In what time could each do it?

59. A and B together can do a piece of work in 12 days, A and C in 15 days, B and C in 20 days. In what time can they do it, all working together?

60. If three pipes can fill a cistern in a , b and c minutes, respectively, in what time will it be filled by all three running together?

NOTE IV.—In questions involving distance, time and rate, we have:

$$(1) \quad \text{Distance} = \text{Time} \times \text{Rate}.$$

$$(2) \quad \text{Time} = \frac{\text{Distance}}{\text{Rate}}.$$

$$(3) \quad \text{Rate} = \frac{\text{Distance}}{\text{Time}}.$$

61. A traveler who goes at the rate of 22 miles in 4 hours, is followed after 5 hours by another, who goes at the rate of 38 miles in 6 hours. In how many hours will the second overtake the first.

Since the first goes 22 miles in 4 hours, his rate per hour is $5\frac{1}{2}$ miles (3). Since the second goes 38 miles in 6 hours, his rate per hour is $6\frac{1}{3}$ miles (3).

Let x = the number of hours the first is traveling,
 Then $x - 5$ = " " " " " second is "
 Then $5\frac{1}{2}x$ = " " " miles the first travels,
 and $6\frac{1}{3}(x - 5)$ = " " " " " second "

They both travel the same distance,

$$\therefore 5\frac{1}{2}x = 6\frac{1}{3}(x - 5).$$

Whence, $x = 38$, Hence, $38 - 5 = 33$, Ans.

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62. A and B depart from the same place to go in the same direction; B travels at the rate of 9 miles in $4\frac{1}{2}$ hours, and A 10 miles in $3\frac{1}{3}$ hours, but B has the start of A by 5 hours; in how many hours will A overtake B?

63. A and B depart at the same time from the same place, to travel in the same direction around an island 36 miles in circumference; A travels 3 miles an hour and B $2\frac{1}{2}$; after how many hours shall they come together?

64. One horse (A) runs at the rate of 1 mile in $2\frac{1}{2}$ minutes, and another (B) at the rate of 1 mile in $3\frac{1}{3}$ minutes; if B has the start of A $\frac{3}{8}$ of a mile how long must A run first, before he overtakes B? Second, before he will be $\frac{1}{4}$ of a mile in advance of B?

65. A man went to town in $6\frac{1}{4}$ hours; in going he traveled at the rate of 6 miles per hour and in returning at the rate of 4 miles per hour. How far was it to town?

66. Two persons set out from the same place in opposite directions. The rate of one of them per hour is 3 miles less than double that of the other, and in 5 hours they are 45 miles apart. Find their rates.

67. A left a certain town at the rate of a miles per hour, and in n hours was followed by B at the rate of b miles per hour. How far did A travel before he was overtaken by B?

68. A hare makes three leaps to the dog's 2, but 3 of the dog's leaps are equal to 7 of the hare's. If the hare has a start of 60 of her own leaps, how many leaps must the hound make to overtake the hare?

Since the hare has the start of 60 of her own leaps we

Let a = the length of a leap of the hare's,

then $\frac{7a}{3}$ = " " " hound.

Let x = the number of leaps of the hound,

then $\frac{3x}{2}$ = " " " hare.

Then, $a\left(60 + \frac{3x}{2}\right)$ = the whole distance.

And $\frac{7ax}{3}$ = " "

$\therefore a\left(60 + \frac{3x}{2}\right) = \frac{7ax}{3}$

Divide by a , $60 + \frac{3x}{2} = \frac{7x}{3}$

Whence, $x = 72$. Ans.

69 A hare has 30 leaps the start of a hound, and makes 6 leaps to the hound's 5, but 7 of the hound's leaps are equal to 9 of the hare's. How many leaps will she make before she is caught?

70. A hound makes 6 leaps while a hare makes 8, but 2 of the hound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 of the hound's leaps. How many leaps does each make before the hare is caught?

In this case, since the distance between the hound and hare is expressed in hound leaps, we let a = the length of a leap of the hound.

71. A hound makes 5 leaps while a hare makes 7, but 8 leaps of the hound are equal to 13 of the hare's. The hare has a start of 72 of the hound's leaps. How many leaps will the hare take before she is caught?

NOTE V.—In problems relating to clocks, it is to be observed:

1°. The minute or long hand moves 12 times as fast as the hour or short hand. Hence, if x = the number of minute-spaces traveled over by the long hand, which will be the case in the following problems, then $\frac{x}{12}$ = the corresponding number traveled over by the short hand.

2°. The difference between x and $\frac{x}{12}$ expresses the distance in minute-spaces that the long hand will gain on the short hand in x minutes. Hence,

$$x - \frac{x}{12} = \text{Distance gained in } x \text{ minutes.}$$

72. Find the time between 3 and 4 o'clock when the hands of a clock:

1°. Are together.

2°. Are at right angles to each other.

3°. Are opposite to each other.

1°. At 3 o'clock the short hand is 15 minute-spaces ahead of the long hand, and when the latter gains this distance on the former, they will be together.

$$\therefore x - \frac{x}{12} = 15.$$

$$\text{Whence, } x = 16\frac{4}{11}.$$

Hence the time is $16\frac{4}{11}$ min. past 3 o'clock.

2°. The hands are at right angles to each other when one is 15 minute-spaces ahead of the other. Hence, they are at right angles at 3 o'clock and will be so again between 3 and 4 o'clock. For, when the long hand gains 15 minute-spaces the hands will be together and when it gains 15 more they will be at right angles. Hence it must gain 30 in all.

$$\therefore x - \frac{x}{12} = 30.$$

$$\text{Whence, } x = 32\frac{8}{11}.$$

Hence, the time is $32\frac{8}{11}$ min. past 3 o'clock.

3°. The hands are opposite when one is 30 minute-spaces ahead of the other. At 3 o'clock the short hand is 15 minutes ahead of the long hand. Hence, it must first gain 15 and then 30, or 45 in all.

$$\therefore x - \frac{x}{12} = 45.$$

$$x = 49\frac{1}{11}.$$

Hence, the time is $49\frac{1}{11}$ min. past 3 o'clock.

73. At what time are the hands of a watch together:

- 1°. Between 5 and 6?
- 2°. Between 8 and 9?
- 3°. Between 11 and 12?

74. At what time are the hands of a watch at right angles:

- 1°. Between 1 and 2?
- 2°. Between 4 and 5?
- 3°. Between 7 and 8?

75. At what time are the hands of a clock opposite to each other:

- 1°. Between 1 and 2?
- 2°. Between 4 and 5?
- 3°. Between 8 and 9?

76. At what time between 7 and 8:

1°, Is the minute hand 5 minutes ahead of the hour hand?

2°, Is the minute hand 10 minutes behind the hour hand.

77. A number of boys have a basket which contains a number of apples; if there were 12 apples more there would be 7 to each boy, and if there were 3 apples less there would be 6 to each boy. Required the number of apples.

78. The denominator of a fraction exceeds its numerator by 3, and if 2 be added to both terms the value of the fraction will be $\frac{3}{4}$. What is the fraction?

79. A and B shoot by turns at a target. A puts 5 bullets out of 7, and B 3 out of 5, into the centre. Between them they put in 92 bullets. How many shots did each fire?

80. I paid \$120 for a horse and sold him at a profit. Had the cost been \$20 less the rate of gain would have been $\frac{1}{4}$ more. What did I receive for the horse?

81. I paid \$40 for a cow and sold her at a loss. Had the cost been less by \$10 the rate of loss would have been less by $\frac{1}{8}$; what did I receive for the cow?

82. A man traveled 10 hours. Had his rate per hour been 2 miles more the time of traveling would have been $2\frac{1}{2}$ hours less; what was his rate per hour?

83. The members of a club subscribe each as many dollars as there are members. If there were 5 more members the total subscription would be \$195 more; how many members are there?

84. The members of a club subscribe each as many dollars as there are members. If there were 12 more members, the subscription from each would be \$10 less, to amount to the same sum. How many members are there?

85. One dray can remove a quantity of freight in 16 loads and another dray can do the same work in 28 loads; if both drays together remove it in 19 loads, how many trips does each dray make?

86. A and B play at a game, agreeing that the loser shall pay to the winner one-third of what he has left after each game. Both together have \$252; in the first game A loses \$99 and in the second B loses \$45, and then each has what he began with. How much had each at first?

87. It is between 5 and 6 o'clock, and the minute hand has passed as far beyond 6 as the hour hand wants of having reached 6. Required the time of day.

88. A and B can perform a piece of work in $5\frac{5}{11}$ days; B and C in $6\frac{2}{3}$ days, and A and C in 6 days: in what time would each of them perform the work alone, and how long would it take them to do the work together?

89. A and B set out from the same place and in the same direction. A travels uniformly 25 miles per day, and after 9 days turns and goes back 15 miles more than B had traveled during those 9 days; he then turns again, and pursuing his journey, overtakes B 30 days after the time they first set out. At what rate did B uniformly travel?

90. A person has a hours at his disposal; how far may he travel in a coach which goes b miles an hour, so as to return home in time, walking back at the rate of c miles per hour?

91. A hound makes m leaps while a hare makes n , but p of the hound's leaps are equivalent to q of the hare's. The hare has a start of h of the hound's leaps; how many leaps does each make before the hare is caught?

92. Suppose a clock to have an hour-hand, a minute-hand and a second-hand, all turning on the same

center. At 3 o'clock the hour-hand is at 3 and the other hands at 12.

1°. How long will it be before the second-hand will be between the other two hands, and at equal distances from each?

2°. Also before the minute-hand will be equally distant between the other two hands?

3°. Also before the hour-hand will be equally distant between the other two hands?

EQUATIONS WITH TWO OR MORE UNKNOWN NUMBERS.

217. Problem. James and Henry together have 12 apples; how many has each?

Let $x =$ the number James has,

and $y =$ " " Henry "

Then $x + y =$ " " both have.

But $12 =$ " " " "

$$\therefore x + y = 12 \dots \dots \dots (1)$$

Now, for every value which we please to ascribe to one of the unknown numbers we can determine the corresponding value of the other, and thus find as many pairs of values as we please, which will satisfy the given equation.

Thus, if $x = 1$, then $1 + y = 12$, $\therefore y = 11$;

if $x = 2$, then $2 + y = 12$, $\therefore y = 10$.

etc., etc., etc.

But suppose we add another condition to the problem, viz.: James has 2 apples more than Henry. This condition gives this additional equation:

$$x - y = 2 \dots \dots \dots (2)$$

Now, we can also find as many pairs of values as we please which will satisfy this equation.

But by combining equations (1) and (2) we can find the only values of x and y which will satisfy both equations, or both conditions of the problem.

Thus, adding (1) and (2), the y disappears, and we obtain $2x = 14$; whence, $x = 7$.

Again, subtracting (2) from (1), the x disappears, and we obtain $2y = 10$; whence $y = 5$.

218. Independent Equations are those which express different relations between the unknown numbers.

Thus, $x + y = 12$ and $5x + 5y = 60$ are not independent equations, for the latter may be obtained by multiplying the former by 5. Hence these equations express the same fact or condition. But $x + y = 12$, and $x - y = 2$ are independent equations, as the one can not be transformed into the other. Hence, these equations express different facts or conditions.

219. Simultaneous Equations are those which are to be satisfied by the same values of the unknown numbers.

220. Elimination is the process of combining simultaneous equations so as to cause the disappearance of one or more of the unknown numbers.

221. There are primarily three methods of elimination, viz.: 1°. By Addition and Subtraction; 2°. By Substitution; 3°. By Comparison.

222. The following is a list of abbreviations used in this book:

1°. To distinguish one equation from another they are numbered thus: (1), (2), ... (n), which numbers are written after the equations and are read, equation 1, equation 2, ... equation n.

2^o. (2) + (3) is read: adding equation 3 to equation 2;

3^o. (2) - (3) " subtracting equation 3 from equation 2;

4^o. (5) $\times m$ " multiplying equation 5 by m ;

5^o. (4) $\div m$ " dividing equation 4 by m ;

and similarly, for other numbers.

223.—ELIMINATION BY ADDITION AND SUBTRACTION.

EXERCISE XLIII.

1. Find the values	$4x + 3y = 22, \dots\dots (1)$
of x and y in:	$5x - 7y = 6, \dots\dots (2)$
(1) $\times 7$,	$28x + 21y = 154, \dots\dots (3)$
(2) $\times 3$,	$15x - 21y = 18, \dots\dots (4)$
(3) + (4),	$43x = 172.$
Whence,	$x = 4.$
Substitute 4 for x in (1),	$16 + 3y = 22.$
Whence,	$y = 2.$

In the preceding solution we eliminated y ; we may also solve it by eliminating x , thus:

(1) $\times 5$,	$20x + 15y = 110, \dots\dots (5)$
(2) $\times 4$,	$20x - 28y = 24, \dots\dots (6)$
(5) - (6)	$43y = 86.$
Whence,	$y = 2.$
Substitute 2 for y in (1),	$4x + 6 = 22.$
Whence,	$x = 4.$

Rule I.—*Multiply the equations by such numbers as will make the coefficients of the quantity to be eliminated equal in the resulting equations.*

II.—*Add the resulting equations, or subtract one from the other according as these equal quantities have unlike or like signs.*

Solve the following:

$$\begin{aligned} 2. \quad 2x + 3y &= 13, \\ 5x + 4y &= 22. \end{aligned}$$

$$\begin{aligned} 4. \quad 2x + 3y &= 70, \\ 4x + 5y &= 130. \end{aligned}$$

$$\begin{aligned} 6. \quad 3x - 5y &= 13, \\ 2x + 7y &= 81. \end{aligned}$$

$$\begin{aligned} 8. \quad 11x + 3y &= 100, \\ 4x - 7y &= 4. \end{aligned}$$

$$\begin{aligned} 3. \quad x &= 4y, \\ 2x + 3y &= 44. \end{aligned}$$

$$\begin{aligned} 5. \quad \frac{x}{2} + \frac{y}{3} &= 7, \\ \frac{x}{3} + \frac{y}{2} &= 8. \end{aligned}$$

$$\begin{aligned} 7. \quad \frac{x}{7} + 7y &= 99, \\ \frac{y}{7} + 7x &= 51. \end{aligned}$$

$$\begin{aligned} 9. \quad 3x + \frac{7y}{2} &= 22, \\ 11y - \frac{2x}{5} &= 20. \end{aligned}$$

$$\begin{aligned} 10. \quad \frac{2x}{3} + \frac{4y}{5} &= 64, \\ \frac{5x}{6} + \frac{9y}{10} &= 77. \end{aligned}$$

224.—ELIMINATION BY SUBSTITUTION.

$$\begin{aligned} 11. \quad \text{Find the values of } x \text{ and } y \text{ in:} \quad & 2x + 3y = 33, \dots\dots (1) \\ & 4x + 5y = 59, \dots\dots (2) \end{aligned}$$

From (1) we find

$$y = \frac{33 - 2x}{3}.$$

$$\text{Substituting this value of } y \text{ in (2), } 4x + 5 \times \left(\frac{33 - 2x}{3} \right) = 59.$$

That is,

$$12x + 165 - 10x = 177.$$

Whence,

$$x = 6.$$

Substituting 6 for x in (1),

$$12 + 3y = 33$$

Whence,

$$y = 7.$$

Rule I.—*From one of the equations find the value of one of the unknown quantities in terms of the other.*

II.—*Substitute this value for the same unknown quantity in the other equation, and reduce the resulting equation.*

Solve the following:

$$\begin{aligned} 12. \quad x + 3y &= 10, \\ 3x + 2y &= 9. \end{aligned}$$

$$13. \quad \frac{x}{5} + \frac{y}{6} = 18,$$

$$\frac{x}{2} - \frac{y}{4} = 21.$$

$$\begin{aligned} 14. \quad 5x + 6y &= 49, \\ 7x - 4y &= 19. \end{aligned}$$

$$15. \quad \frac{x}{3} + \frac{y}{4} = 9,$$

$$\frac{x}{4} + \frac{y}{5} = 7.$$

$$\begin{aligned} 16. \quad 3x - 2y &= 1, \\ 3y - 4x &= 1. \end{aligned}$$

$$\begin{aligned} 17. \quad 3x + 4y &= 88, \\ 6x + 5y &= 61. \end{aligned}$$

$$18. \quad \frac{x}{3} + \frac{y}{4} = 1,$$

$$\frac{x}{2} + \frac{y}{3} = 1.$$

$$19. \quad \frac{x+y}{2} - \frac{x-y}{3} = 8, \quad \frac{x+y}{3} + \frac{x-y}{4} = 11.$$

225.—ELIMINATION BY COMPARISON.

$$\begin{aligned} 20. \text{ Find the values} \quad 3y + 2x &= 31 \dots\dots (1) \\ \text{of } x \text{ and } y \text{ in:} \quad 5y + 4x &= 55 \dots\dots (2) \end{aligned}$$

$$\text{From (1) we find} \quad y = \frac{31 - 2x}{3} \dots\dots (3)$$

$$\text{From (2) we find} \quad y = \frac{55 - 4x}{5} \dots\dots (4)$$

Equating these values of y , we have

$$\frac{31 - 2x}{3} = \frac{55 - 4x}{5}$$

Whence, $x = 5$

Substituting 5 for x in (3), we find $y = 7$.

Rule I.—*From each equation find the value of one of the unknown quantities in terms of the other.*

II.—*Equate the values thus obtained, and reduce the resulting equation.*

Solve the following:

$$\begin{aligned} 21. \quad 3y - 7x &= 4, \\ 2y + 5x &= 22. \end{aligned}$$

$$22. \quad \frac{1}{3}x - \frac{1}{5}y = 4,$$

$$\frac{1}{7}x + \frac{1}{3}y = 8.$$

$$\begin{aligned} 23. \quad 5x + 7y &= 43, \\ 11x + 9y &= 69. \end{aligned}$$

$$24. \quad \frac{3x}{4} + \frac{2y}{3} = 15,$$

$$\frac{5x}{6} + \frac{7y}{9} = 17.$$

$$\begin{aligned} 25. \quad 11x - 10y &= 14, \\ 5x + 7y &= 41. \end{aligned}$$

$$\begin{aligned} 26. \quad 30x - 77y &= -295, \\ 20x + 21y &= 165. \end{aligned}$$

$$\begin{aligned} 27. \quad 8x - 5y &= 1, \\ \frac{11x - 5y}{22} &= \frac{3x + y}{32}. \end{aligned}$$

$$\begin{aligned} 28. \quad 11x - 13y &= 0, \\ 13x - 11y &= 48. \end{aligned}$$

$$29. \quad \frac{x}{8} + \frac{y}{9} = 42,$$

$$\frac{x}{9} + \frac{y}{8} = 43.$$

NOTE.—The following may be solved by either of the preceding methods.

$$30. \quad \frac{x+y}{5} - \frac{x-y}{2} = 3, \quad 31. \quad \begin{aligned} 4.5x + .25y &= 10, \\ .75x - .15y &= .9. \end{aligned}$$

$$\frac{x-y}{2} + \frac{x+y}{10} = 9.$$

$$32. \quad \frac{3x}{10} - \frac{y}{15} - \frac{4}{9} = \frac{x}{12} - \frac{y}{18},$$

$$2x - \frac{8}{3} = \frac{x}{12} - \frac{y}{15} + \frac{11}{10}.$$

$$\begin{aligned} 33. \quad 2x + .5y &= 1.8, \\ .5x - .8y &= .08. \end{aligned}$$

$$34. \quad 2x - \frac{y+3}{4} = 8,$$

$$4y - \frac{8-x}{3} = 24\frac{1}{2} - \frac{2y+1}{2}.$$

$$\begin{aligned} 35. \quad x + y &= s, \\ x - y &= d. \end{aligned}$$

$$\begin{aligned} 36. \quad 3x + 2y &= a, \\ 4x - 3y &= b. \end{aligned}$$

$$\begin{aligned} 37. \quad \frac{x-2}{5} - \frac{10-x}{3} &= \frac{y-10}{4}, \\ \frac{2y+4}{3} - \frac{2x+y}{8} &= \frac{x+13}{4}. \end{aligned}$$

$$\begin{aligned} 38. \quad ax + by &= m, \dots\dots (1) \\ cx + dy &= n, \dots\dots (2) \end{aligned}$$

$$\begin{aligned} (1) \times c, & \quad acx + bcy = cm \dots\dots\dots (3) \\ (2) \times a, & \quad acx + ady = an \dots\dots\dots (4) \\ (3) - (4), & \quad (bc - ad)y = cm - an \end{aligned}$$

Whence,

$$y = \frac{cm - an}{bc - ad}.$$

To find x :

$$\begin{aligned} (1) \times d, & \quad adx + bdy = dm \dots\dots\dots (5) \\ (2) \times b, & \quad bdx + bdy = bn \dots\dots\dots (6) \\ (5) - (6), & \quad (ad - bc)x = dm - bn \end{aligned}$$

Whence,

$$x = \frac{dm - bn}{ad - bc}.$$

$$\begin{aligned} 39. \quad mx - ny &= r, \\ px - qy &= s. \end{aligned}$$

$$\begin{aligned} 40. \quad ax + by &= c, \\ mx - ny &= o. \end{aligned}$$

$$41. \quad \frac{1}{x} + \frac{1}{y} = a,$$

$$42. \quad \frac{a}{x} + \frac{b}{y} = c,$$

$$\frac{1}{x} - \frac{1}{y} = b.$$

$$\frac{a}{x} - \frac{b}{y} = d.$$

$$43. \quad \frac{x}{a+b} + \frac{y}{a-b} = 2a, \quad \frac{x-y}{4ab} = 1.$$

226. If there are three equations, (1), (2), (3), involving three unknown quantities,

1°. Combine two of the equations, as (1) and (2), so as to eliminate one of the unknown quantities, and call the resulting equation (6).

2°. Combine the remaining equation, (3), with either of the other two, as (1) or (2), so as to eliminate the same unknown quantity, and call the resulting equation (9).

3°. Combine (6) and (9) so as to find the unknown quantities which they involve.

4°. Substitute the values thus found in (1). (2) or (3), and thus find the remaining unknown quantity.

EXERCISE XLIV.

1. Find the values of x , y and z in:
- $$2x + 3y + 4z = 16 \dots (1)$$
- $$3x + 2y - 5z = 8 \dots (2)$$
- $$5x - 6y + 3z = 6 \dots (3)$$

NOTE.—It is immaterial which unknown quantity we eliminate first, but there will be less work in eliminating the one which requires the least number of multiplications or the smallest multipliers to make its coefficients equal. For instance, it is best to eliminate y first in the preceding example.

$$(1) \times 2 \qquad 4x + 6y + 8z = 32 \dots (4)$$

$$(3) \text{ is } \qquad 5x - 6y + 3z = 6 \dots (5)$$

$$(4) + (5) \qquad 9x \qquad + 11z = 38 \dots (6)$$

$$(2) \times 3 \qquad 9x + 6y - 15z = 24 \dots (7)$$

$$(3) \text{ is } \qquad 5x - 6y + 3z = 6 \dots (8)$$

$$(7) + (3) \qquad 14x \qquad - 12z = 30 \dots (9)$$

$$(6) \times 14 \qquad 126x \qquad + 154z = 532 \dots (10)$$

$$(9) \times 9 \qquad 126x \qquad - 108z = 270 \dots (11)$$

$$(10) - (11) \qquad 262z = 262$$

$$\therefore z = 1$$

Substituting 1 for z in (11), we find $x = 3$

Substituting 1 for z and 3 for x in (1), we have

$$6 + 3y + 4 = 16; \text{ whence, } y = 2.$$

227. If four or more equations are given, involving four or more unknown quantities, we proceed in a similar manner. Thus: Eliminate one of the unknown quantities between three or more pairs of the equations; then a second between the different pairs of the resulting equations; and so on.

Solve the following:

2. $2x - y + z = 9,$
 $x - 2y + 3z = 14,$
 $3x + 4y - 2z = 7.$
3. $4x - 3y + 2z = 40,$
 $5x + 9y - 7z = 47,$
 $9x + 8y - 3z = 97.$
4. $2x - 3y + 5z = 15,$
 $3x + 2y - z = 8,$
 $-x + 5y + 2z = 21.$
5. $5x - 6y + 4z = 15,$
 $7x + 4y - 3z = 19,$
 $2x + y + 6z = 46.$
6. $2x - 4y + 9z = 28,$
 $7x + 3y - 5z = 3,$
 $9x + 10y - 11z = 4.$
7. $2x + 3y - 4z = 10,$
 $-4y + 5z + 3x = 14,$
 $6z + 4x - 5y = 18.$
8. $3x - 3y + z = 0,$
 $2x - 7y + 4z = 0,$
 $9x + 5y + 3z = 28.$
9. $x + y + z = 29,$
 $x + 2y + 3z = 62,$
 $\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 10.$
10. $7x - 3y = 1,$
 $11z - 7u = 1,$
 $4z - 7y = 1,$
 $19x - 3u = 1.$
11. $x + y + z + u = 14,$
 $2x + 3y + 4z + 5u = 54,$
 $4x - 5y - 7z + 9u = 10,$
 $3x + 4y + 2z - 3u = 11.$
12. $ay + bx = c,$
 $cx + az = b,$
 $bz + cy = a.$
13. $ax + by + cz = n,$
 $a^2x + b^2y + c^2z = n^2,$
 $a^3x + b^3y + c^3z = n^3.$
14. $bx + ay = ab,$
 $cx + az = ac,$
 $cy + bz = bc.$
15. $x + y + z = a + b + c,$
 $bx + cy + az = a^2 + b^2 + c^2,$
 $cx + ay + bz = a^2 + b^2 + c^2.$

PROBLEMS.

EXERCISE XLV.

228.—1. Find two numbers such that if the first be added to 4 times the second, the sum is 29; and if the second be added to 6 times the first, the sum is 36.

Let x = the first number, and y = the second.

Then $x + 4y = 29 \dots\dots\dots (1)$

and $6x + y = 36 \dots\dots\dots (2)$

(1) $\times 6$, $6x + 24y = 174 \dots\dots\dots (3)$

(3) $-$ (2), $23y = 138$

$y = 6.$

Substituting 6 for y in (2), we have

$$6x + 6 = 36. \text{ Whence, } x = 5.$$

2 Find two numbers such that 2 times the first plus the second is 17, and 2 times the second plus the first is 19.

3. Find two numbers such that 4 times the first less 1 is equal to 3 times the second plus 2; and 3 times the first plus 12 is equal to 4 times the second less 12.

4. Divide 100 into two parts such that 10 shall be the difference between the first part and 5 times the second part.

5. A and B together have \$570. If A had 3 times as much as he has, and B 5 times as much as he has, the sum would be \$2350. How much has each?

6. If A's money were increased by \$36 he would have 3 times as much as B; but if B's money were diminished by \$5 he would have $\frac{1}{2}$ as much as A. How much has each?

N. E. A:—9.

7. Find two numbers such that the sum of 4 times the first and $\frac{1}{3}$ of the second is 53; and the sum of 3 times the second and $\frac{1}{4}$ of the first is 48.

Let x = the first and y = the second.

Then, $4x + \frac{1}{3}y = 53 \dots\dots\dots (1)$

$\frac{1}{4}x + 3y = 48 \dots\dots\dots (2)$

$(1) \times 9, \quad 36x + 3y = 477 \dots\dots\dots (3)$

$(3) - (2), \quad 35\frac{3}{4}x = 429 \dots\dots\dots (4)$

$(4) \times 4, \quad 143x = 1716$

$\therefore x = 12.$

Substituting 12 for x in (2), we find $y = 15$.

8. Find two numbers such that 7 times the first increased by $\frac{1}{8}$ of the second shall be 144; and 10 times the second increased by $\frac{1}{5}$ of the first shall be 324.

9. Find two numbers such that 3 times the first diminished by $\frac{1}{5}$ of the second shall be 121; and 2 times the second diminished by $\frac{1}{3}$ of the first shall be 76.

10. Divide 100 into two parts such that 3 times the first part diminished by $\frac{1}{6}$ of the second shall be 110.

11. If A had 6 times as much money as he has he would lack $\frac{1}{7}$ of what B has of having \$1100; and if B had 7 times as much as he has he would lack $\frac{1}{5}$ of what A has of having \$1000. How much has each?

12. Find two numbers such that $\frac{1}{2}$ of the first and $\frac{3}{4}$ of the second together may be equal to the difference of 3 times the first and the second, and this difference equal to 11.

13. Divide 9 into three parts such that the sum of the first, 2 times the second, and 3 times the third shall be 22; and the sum of the first, 4 times the second, and 9 times the third, 58.

14. A certain fraction becomes equal to $\frac{1}{2}$ when 1 is added to its numerator, and equal to $\frac{1}{3}$ when 1 is added to its denominator. What is the fraction?

Let $\frac{x}{y}$ = the required fraction.

Then, $\frac{x+1}{y} = \frac{1}{2}$

and $\frac{x}{y+1} = \frac{1}{3}$.

From the solution of these equations it is found that $x = 3$, and $y = 8$.

Therefore the fraction is $\frac{3}{8}$.

15. If the numerator of a fraction is decreased by 2 its value becomes $\frac{1}{4}$; if the denominator is increased by 3 its value becomes $\frac{1}{5}$. What is the fraction?

16. What fraction is that whose numerator being doubled, and the denominator increased by 7, becomes $\frac{2}{3}$; but, the denominator being doubled, and the numerator increased by 2, becomes $\frac{3}{5}$?

17. A fraction which is equal to $\frac{3}{4}$, becomes $\frac{2}{3}$ when a certain number is subtracted from both of its terms; and $\frac{4}{5}$, when the same number is added to its numerator and 5 to its denominator. What is the fraction?

18. The numerator of one fraction is 1, and the denominator of another is 3; what are the fractions if their sum is $\frac{5}{6}$, and their difference, $\frac{1}{2}$?

19. Find two fractions whose numerators are 1 and 2, whose sum is $\frac{7}{12}$, and whose sum will be $\frac{2}{3}$ if the denominators are interchanged?

NOTE.—If x and y represent the tens' and units' digits of a number respectively, then the number itself is represented by $10x + y$. Likewise, a number consisting of three digits may

be represented by $100x + 10y + z$, in which x , y and z represent the digits of the number.

Thus, 753 means $700 + 50 + 3$, or 100 times 7 + 10 times 5 + 3. Here, 7, 5 and 3 are the digits of the number 753.

20. The sum of the two digits of a number is 6, and if 18 be added to the number the digits will be interchanged. What is the number?

Let x = the digit in the tens' place.

and y = the digit in the units' place.

Then $10x + y$ = the number.

By the conditions, $x + y = 6$(1)

and $10x + y + 18 = 10y + x$(2)

From (2) $9x - 9y = -18$(3)

(3) $\div 9$ $x - y = -2$(4)

(1) + (4), $2x = 4, \therefore x = 2$

(1) - (4), $2y = 8, \therefore y = 4$.

Hence, the number is 24.

21. A number expressed by two digits is 4 times the sum of the digits, and if 27 be added to the number the digits will be interchanged; find the number?

22. A number of two digits is 4 times the sum of its digits, and 3 times the number is equal to the square of the sum of its digits. What is the number?

23. A number is expressed by three digits. The middle digit is twice the left-hand digit, and one less than the right-hand digit. If 297 be added to the number, the order of the digits will be reversed; find the number.

24. A number is expressed by three digits. The sum of the digits is 18; the number is equal to 99 times the sum of the first and third digits, and if 693 be subtracted from the number the order of the digits will be reversed. Find the number.

25. If a certain number be divided by the sum of its two digits the quotient is 3, and the remainder 3; if the digits be interchanged, and the resulting number be divided by the sum of the digits, the quotient is 7, and the remainder 9. What is the number?

26. The sum of the three digits of a number is 16. If the number be divided by the sum of its hundreds' and units' digits the quotient is 77 and the remainder 6; and if it be divided by the number expressed by its two right-hand digits, the quotient is 16 and the remainder 5. Find the number.

27. The sum of the two digits of a number is 8, and if $9d$ be added to the number, the order of the digits will be reversed. What is the number?

28. A person has two horses, and a carriage which is worth \$150. The value of the first horse and carriage is 2 times that of the second horse, and the value of the second horse and carriage is 3 times that of the first horse. What is the value of each horse?

29. A person has two horses, and a carriage which is worth \$ c . The first horse and the carriage are worth m times the second horse, and the second horse and carriage are worth n times the first horse. What is the value of each horse?

30. Five years ago the age of a father was 7 times that of his son; five years hence the age of the father will be 3 times that of the son. What are their ages?

31. Two years hence A will be twice as old as B was 6 years since, and 8 years hence B will be twice as old as A was 15 years since. What are their ages?

32. If A give B \$5, they will have equal sums of money; but if B give A \$5, then 7 times A's money will be equal to 11 times B's. How much has each?

33. If A give B \$ a , A will have twice as much as B; and if B give A \$ b , A will lack \$ c of having 4 times as much as B. How much has each?

34. A bought 5 hogs and 7 sheep for \$47 $\frac{1}{4}$, and B bought of the same lot 9 hogs and 4 sheep for \$48 $\frac{1}{2}$; what was the price of the hogs and sheep?

35. A pound of tea and 3 pounds of sugar cost 6 shillings, but if sugar were to rise 50 per cent. and tea 10 per cent. they would cost 7 shillings. Find the price of tea and sugar.

36. A grocer has two kinds of tea which are worth 40 cents and 72 cents a pound. How much of each must he take to make a mixture of 50 pounds worth 60 cents a pound?

37. A wine-merchant has two kinds of wine which are worth 90 cents and 28 cents a quart, out of which he forms a mixture which is worth \$61.20. He sells the mixture at 50 cents a quart and gains \$3.80. How many quarts of each did he put into the mixture?

38. A and B can do a certain work in 24 days, B and C in 40 days, A and C in 30 days. In how many days can each do the work?

Let x = the number of days it will take A to do the work.
 y = " " " B "
 z = " " " C "

Then, $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, respectively, will denote the part each can do in one day.

$$\begin{array}{lcl}
 \text{Hence,} & \frac{1}{x} + \frac{1}{y} = \frac{1}{24} = \frac{5}{120} & \dots\dots\dots (1) \\
 & \frac{1}{y} + \frac{1}{z} = \frac{1}{40} = \frac{3}{120} & \dots\dots\dots (2) \\
 & \frac{1}{x} + \frac{1}{z} = \frac{1}{30} = \frac{4}{120} & \dots\dots\dots (3) \\
 (1) + (2) + (3), & \frac{2}{x} + \frac{2}{y} + \frac{2}{z} = \frac{12}{120} & \dots\dots\dots (4) \\
 (1) \times 2, & \frac{2}{x} + \frac{2}{y} = \frac{10}{120} & \dots\dots\dots (5) \\
 (4) - (5), & \frac{2}{z} = \frac{2}{120}; \text{ whence, } z = 120. & \\
 (4) - 2 \times (2), & \frac{2}{x} = \frac{6}{120}; \text{ whence, } x = 40. & \\
 (4) - 2 \times (3) & \frac{2}{y} = \frac{4}{120}; \text{ whence, } y = 60. &
 \end{array}$$

39. A and B can do a certain work in $4\frac{1}{2}$ days, A and C in 6 days, and B and C in 8 days. In how many days can each do the work?

40. A box will hold 20 spellers and 15 readers; and 8 spellers and 12 readers will fill $\frac{3}{8}$ of it. How many of each will it hold?

41. A had enough money to buy 10 hats and 12 caps; after buying 8 hats and 4 caps he observed that he had spent $\frac{5}{8}$ of his money. How many of each could he have bought with all his money?

42. A, with the help of B for 4 days, can do a certain work in 15 days; B, with the help of C for 6 days, in 20 days; C, with the help of A for 7 days, in 22 days. How long will it take each alone to do the work?

43. A cistern has three pipes A, B, and C. A and B will fill it in a minutes; A and C in b minutes; B and C in c minutes. How long will it take each alone to fill it?

44. A can row a boat down stream at the rate of 9 miles per hour, and up stream at the rate of 3 miles per hour. At what rate does the stream flow, and at what rate could A row the boat in still water?

Let x = A's rate per hour in still water,
 and y = the rate per hour of the stream.
 Then $x + y$ = A's rate per hour down stream,
 and $x - y$ = " " " " up "
 $\therefore x + y = 9,$
 $x - y = 3,$
 Whence, $x = 6, y = 3.$

45. A crew which can pull at the rate of 15 miles an hour down stream, finds that it takes 3 times as long to come up the river as to go down. At what rate does the stream flow?

46. A man sculls down a stream, which runs at the rate of 3 miles an hour, for a certain distance in 3 hours, and finds that it takes him 12 hours to return. Find the distance he pulled down the stream and the rate of his pulling.

47. A boatman can row down stream a distance of a miles and back again in b hours; and he finds that he can row d miles against the current in the same time he rows c miles with it. Required the time in going and returning, also the rate of the current.

48. Find three numbers, so that the first with half the other two, the second with one-third the other two, and the third with one-fourth the other two, shall each be equal to 34.

49. A says to B, "give me $\frac{1}{2}$ of your money and I shall lack \$52 of having twice as much as you;" B says to A, "give me $\frac{1}{4}$ of your money and I shall lack

\$40 of having thrice as much as you." How much has each?

50. If the length and width of a rectangular field were increased by 5 rods and 4 rods respectively, its area would be increased by 240 square rods; but if the length and width were diminished by 4 rods and 5 rods respectively, its area would be diminished by 210 square rods. Required, its length, width and area.

51. A and B run a mile. First A gives B a start of 44 yards and beats him by 51 seconds; at the second heat A gives B a start of 1 minute and 15 seconds, and is beaten by 88 yards. Find the times in which A and B can run a mile separately.

52. The fore-wheel of a carriage makes six revolutions more than the hind-wheel in going 120 yards; if the circumference of the fore-wheel be increased by $\frac{1}{4}$ of its present size, and the circumference of the hind-wheel by $\frac{1}{5}$ of its present size, the six will be changed to four. Required the circumference of each wheel.

53. A railway train after traveling for one hour meets with an accident which delays it one hour, after which it proceeds at $\frac{3}{5}$ of its former rate, and arrives at the terminus 3 hours behind time; had the accident occurred 50 miles further on, the train would have arrived 1 hour and 20 minutes sooner. Required the length of the line.

54. A's capital is invested at a certain per cent. B's capital is \$1000 more, and he receives one per cent. more than A; C's capital is \$1500 more, and he receives two per cent. more than A. B's income is \$80 greater than A's, and C's income is \$150 greater than

A's. Find the capital of each, and the rate at which it is invested.

55. If A give B as much money as B has, then B give A as much as A has, then A give B as much as B has, then B give A as much as A has, each would have \$16. How much has each?

56. A merchant has three casks, A, B and C, each containing wine. If he pour from A into B and C each as much as it contains, then pour from B into A and C each as much as it contains, then pour from C into A and B each as much as it contains, each would contain 16 gallons. How many gallons in each?

INDETERMINATE EQUATIONS.

229. An **Indeterminate Equation** is one in which the unknown numbers have an indefinite number of values.

230. I.—*One equation containing two or more unknown numbers is indeterminate.*

Thus, take the equation $3x - 4y = 12 \dots \dots (1)$

By reducing, we get $x = \frac{12 + 4y}{3}$

Let $y = 1, 2, 3, 4, 5$, etc.

Then $x = 5\frac{1}{3}, 6\frac{2}{3}, 8, 9\frac{1}{3}, 11\frac{2}{3}$, etc.

Any two corresponding values of x and y will satisfy (1).

231. II.—*Simultaneous equations are indeterminate when the number of unknown numbers involved exceeds the number of independent equations.*

For by eliminating, the equations may be reduced to

one equation involving two or more unknown numbers, which is indeterminate, as before shown.

Thus, take these equations:

$$3x + 4y - 6z = 40 \dots\dots\dots (1)$$

$$5x - 3y + 2z = 25 \dots\dots\dots (2)$$

Here we have *two* equations involving *three* unknown numbers.

Multiply (2) by 3, add the resulting equation to (1), and we obtain

$$18x - 5y = 115,$$

which is indeterminate.

232. Generally, the object in solving indeterminate equations and problems, is to find only the positive integral values of the unknown quantities.

EXERCISE XLVI.

1. Solve in positive integers $4x + 9y = 115 \dots (1)$

Transpose, etc., $x = \frac{115 - 9y}{4} = 28 - 2y + \frac{3 - y}{4} \dots (2)$

$$\therefore x - 28 + 2y = \frac{3 - y}{4} \dots\dots\dots (3)$$

Since the values of x and y are to be integral, $x - 28 + 2y$ will be integral, and hence, $\frac{3 - y}{4}$ will be integral, otherwise we should have an integer equal to a fraction. Hence, we may write

$$\frac{3 - y}{4} = n, \text{ an integer.}$$

Then,

$$3 - y = 4n, \text{ or } y = 3 - 4n \dots\dots (4)$$

Substitute in (2),

$$x = 22 + 9n \dots\dots (5)$$

In (4) and (5) n may have any integral value, except such as produce negative values of x or y .

Hence, in (4), n can have no positive value greater than 0, but may have any negative value.

But we see from (5) that if the absolute value of n exceeds 2 (n being negative), x will be negative.

Therefore, n can have but three values: 0, -1, -2.

Substituting these values in (4) and (5), we have

$$y = 3, \quad 7, \quad 11;$$

$$x = 22, \quad 13, \quad 4.$$

233. In solving equations like the preceding it is desirable that the coefficient of y should be *one* more than some multiple of the coefficient of x , as in (1). If this relation does not exist, the equation should be multiplied by such a number as will make the new coefficient of y *one* more than some multiple of the original coefficient of x , and then proceed as in the solution of the next example.

2. Solve in positive integers $5x - 14y = 11 \dots (1)$

Here, $14 \div 5 = 2$ quo. and 4 rem., which is not in the desired form.

Multiply (1) by 4, and we have $4 \times 5x - 56y = 44 \dots \dots (2)$

Now, $56 \div 5 = 11$ quo. and 1 rem., which is in the desired form. It is useless to consider 4, as it is a factor of all the terms.

Transpose, divide by 5,

$$4x = \frac{44 + 56y}{5} = 8 + 11y + \frac{4 + y}{5} \dots \dots (3)$$

Make $\frac{4 + y}{5} = n$, whence, $y = 5n - 4 \dots \dots (4)$

Substitute in (3) $x = 14n - 9 \dots \dots (5)$

Here it is evident that n may have any positive value.

Making $n = 1, 2, 3$, etc.,

We have $y = 1, 6, 11$, etc.,

and $x = 5, 19, 23$, etc.

Solve in positive integers:

- | | |
|-----------------------|------------------------|
| 3. $3x + 7y = 29.$ | 4. $4x - 7y = 13.$ |
| 5. $5x + 11y = 61.$ | 6. $6x - 5y = 20.$ |
| 7. $7x + 12y = 88.$ | 8. $13x - 9y = 103.$ |
| 9. $x + 13y = 28.$ | 10. $5x - 13y = 1.$ |
| 11. $23x + 17y = 47.$ | 12. $43x - 35y = -19.$ |

234. An Indeterminate Problem is one which admits of an indefinite number of solutions.

13. What number is it, which, when divided by 13 the remainder is 8; and when divided by 17 the remainder is 5?

Let m denote the number, then

$$\frac{m-8}{13} = x, \quad \text{and} \quad \frac{m-5}{17} = y.$$

Whence, $m = 13x + 8, \dots (1)$ and $m = 17y + 5 \dots (2)$

$\therefore \quad \quad \quad 13x + 8 = 17y + 5 \dots (3)$

$\quad \quad \quad 13x = 17y - 3 \dots (4)$

$(4) \times 10, \quad \quad \quad 10 \times 13x = 170y - 30 \dots \dots \dots (5)$

$(5) \div 13 \quad \quad \quad 10x = 13y - 2 + \frac{y-4}{13} \dots \dots \dots 6)$

Make $\frac{y-4}{13} = n$, whence, $y = 13n + 4$.

Substitute in (2) $m = 221n + 73$.

In this n may have any positive value; hence, the number is 73, or 294, or 515, etc.

14. The sum of three positive integers is 11; and if the first be multiplied by 3, the second by 5, and the third by 7, the sum of the products will be 57. What are the numbers?

Let $x =$ the first, $y =$ the second, $z =$ the third.

Then $x + y + z = 11 \dots \dots \dots (1)$

and $3x + 5y + 7z = 57 \dots \dots \dots (2)$

$(1) \times 7 - (2), \quad \quad \quad 4x + 2y = 20,$

$\quad \quad \quad 2x + y = 10,$

$\quad \quad \quad y = 10 - 2x.$

Here x may have any value from 1 to 4 inclusive.

Making $x = 1, 2, 3, 4;$

then $y = 8, 6, 4, 2;$

and $z = 2, 3, 4, 5.$

15. Divide 100 into two parts, such that if one of them be divided by 6 and the other by 13, the respective remainders shall be 5 and 7.

16. Can the equation $6x + 9y = 40$ be solved in positive integers?

17. Find the least number which, being divided by 7, leaves a remainder 2, and being divided by 5, leaves a remainder 3.

18. How many pairs of fractions are there with denominators 7 and 5 whose sum is $\frac{67}{35}$?

19. Two fractions are equal to $\frac{2}{3}$ and $\frac{1}{12}$, respectively, and the sum of their numerators is 49. What are the fractions?

20. A number that is a multiple of 5, when increased by 2, is a multiple of 14. What is the number?

21. The numerator of a fraction is a multiple of 2, the denominator a multiple of 3, and if 5 be added to both terms, the value of the fraction will be $\frac{3}{4}$. Find the fraction.

22. What is the least possible number of oranges on a tree, if in counting them 11 at a time there are 3 over; and in counting them 7 at a time there are 2 over?

23. Divide 70 into three parts which shall be divisible by 6, 7, 8, respectively, and the sum of the quotients shall be 10.

24. If a number consisting of two digits be divided by the sum of its digits, the quotient will be 8 and the remainder 5. Find the number.

25. Forty-eight stones are placed in a straight line and extend over a distance of 175 feet; some of them are 3 ft., some 5 ft., and others 8 ft. in length. How many of each are there?

26. A man has a number of \$10, \$5, \$1 pieces, how many of each must he take to pay a debt of \$120, using forty pieces in all?

27. A farmer buys oxen, hogs and hens. The whole number bought is 100, and the whole cost \$100. If the oxen cost \$10, the hogs \$2½ and the hens \$½, each, how many of each did he buy?

28. Forty-one persons, men, women and children, spent 40s., whereof each man paid 4s., each woman 3s., and each child 4d. How many of each were there?

29. A, B and C had 10, 20 and 30 oranges, respectively, which they sold, in two sales, so that all received equal sums for all their oranges. In the first sale each received 2 cents per orange, and in the second, 7 cents per orange. How many did each sell at the first sale, and how much did each receive for all his oranges?

INCOMPATIBLE EQUATIONS.

235. **Incompatible Equations** are those which can not be satisfied for the same values of the unknown numbers.

236. *If the number of independent equations exceeds the number of unknown numbers, the equations may be incompatible.*

Thus, take the equations

$$x + y = 12 \dots\dots\dots (1)$$

$$x - y = 2 \dots\dots\dots (2)$$

$$x - 2y = 0 \dots\dots\dots (3)$$

1°. Combining (1) and (2), we find $x = 7$ and $y = 5$; but these values will not satisfy (3).

2°. Combining (1) and (3), we find $x = 8$ and $y = 4$; but these values will not satisfy (2).

3°. Combining (2) and (3), we find $x = 4$ and $y = 2$, but these values will not satisfy (1).

EXERCISE XLVII.

1. Show that $\left\{ \begin{array}{l} 2x + 3y = 8, \\ 3x + 7y = 7, \\ x - 6y = 19, \end{array} \right\}$ are incompatible.

237. Incompatible Conditions are those which may be represented by incompatible equations.

238. Impossible Problems are those which involve incompatible conditions.

NOTE.—Let the student show that the following are impossible problems. To do so, it will be sufficient to show that the equations representing the conditions are incompatible.

2. Divide \$75 between A and B so that if B give A \$5, A will then have 2 times as much as B; but if A give B \$5, they will have equal sums.

3. A line 96 feet long is cut into two parts whose lengths are proportional to 5 and 3. What are the lengths of the parts, if one part is 10 ft. longer than the other?

4. A and B buy 200 acres of land at \$2 per acre, each paying \$200. If A takes the better portion of the land at \$2½ per acre and B the remainder at \$1¾ per acre, how much of the land will each receive?

CHAPTER VI.

HIGHER OPERATIONS.

INVOLUTION.

239. Any power of a number may be expressed by giving it the exponent of the required power.

Thus, the third power of $5a^2b$ is $(5a^2b)^3$;

the fourth power of $\frac{2ab}{3n}$ is $\left(\frac{2ab}{3n}\right)^4$;

the n th power of $2a + b$ is $(2a + b)^n$.

POWERS OF MONOMIALS.

240. It follows from Art. 59, that

$$(4a^3b^2)^3 = 4a^3b^2 \times 4a^3b^2 \times 4a^3b^2 = 4^3a^9b^6 = 64a^9b^6.$$

Hence, to find any power of a monomial, *multiply the exponent of each factor by the exponent of the required power.*

The coefficient, if any, may then be simplified as in the above example.

241. It follows from Art. 192, that,

$$\left(\frac{2a^2b}{3c}\right)^3 = \frac{2a^2b}{3c} \times \frac{2a^2b}{3c} \times \frac{2a^2b}{3c} = \frac{8a^6b^3}{27c^3}.$$

Hence, to find any power of a fraction, *raise the terms separately to the required power.*

242. It is evident that (60),

1°. *All even powers of a number are positive.*

2°. *All odd powers of a number have the same sign as the number itself.*

EXERCISE XLVIII.

Write the second members of the following equations :

1. $(a^2)^2 =$ 2. $(3a^2c)^5 =$ 3. $(-7a^3bz^2)^3 =$
4. $(a^4)^3 =$ 5. $(-2x^3y)^4 =$ 6. $(-4b^2c^n)^4 =$
7. $(a^n)^4 =$ 8. $(-3x^2z)^3 =$ 9. $(-3a^n x^3y)^5 =$
10. $(a^2b^3)^5 =$ 11. $(5a^4b^2)^2 =$ 12. $(2a^m b^n c)^6 =$
13. $\left(\frac{a^2b}{3}\right)^4 =$ 14. $\left(\frac{2x^2y}{5z^3}\right)^3 =$ 15. $\left(-\frac{5a^2b}{2c^4}\right)^4 =$
16. $\left(\frac{3xy^3}{4}\right)^5 =$ 17. $\left(-\frac{m^3}{n^2x}\right)^7 =$ 18. $\left(-\frac{a^n b^2}{c^m}\right)^6 =$

POWERS OF BINOMIALS.

243. By actual multiplication we find that :

$$(a + b)^1 = a + b.$$

$$(a + b)^2 = a^2 + 2ab + b^2.$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

From these results it will be observed that :

I. The number of terms is greater by one than the exponent of the power to which the binomial is raised.

II. In the first term, the exponent of a is the same as the exponent of the power to which the binomial is

raised; and it decreases by one in each succeeding term.

III. b appears in the second term with 1 for an exponent, and its exponent increases by one in each succeeding term.

IV. The coefficient of the first term is 1.

V. The coefficient of the second term is the same as the exponent of the power to which the binomial is raised.

VI. The coefficient of each succeeding term is found from the next preceding term by multiplying its coefficient by the exponent of a , and dividing the product by a number greater by one than the exponent of b .

244. If b be negative, the terms in which the odd powers of b occur are negative. Thus:

$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

EXERCISE XLIX.

Write by inspection the results:

- | | | |
|-------------------|-------------------|-------------------|
| 1. $(a + x)^3 =$ | 2. $(a + x)^4 =$ | 3. $(x + y)^5 =$ |
| 4. $(a - x)^3 =$ | 5. $(x - a)^4 =$ | 6. $(x - y)^5 =$ |
| 7. $(a + 1)^3 =$ | 8. $(x + 1)^4 =$ | 9. $(x + 1)^5 =$ |
| 10. $(x - 1)^3 =$ | 11. $(x - 1)^4 =$ | 12. $(y - 1)^5 =$ |

245. The same method may be employed when the terms of a binomial have coefficients or exponents.

Let it be observed that the following results are in accordance with the preceding rule, provided, that a be called "the first term of the binomial," and b , "the second term of the binomial."

$$(3x - y^2)^3 = (3x)^3 - 3(3x)^2(y^2) + 3(3x)(y^2)^2 - (y^2)^3 \\ = 27x^3 - 27x^2y^2 + 9xy^4 - y^6.$$

$$(c^2 - \frac{1}{2}x)^4 = (c^2)^4 - 4(c^2)^3(\frac{1}{2}x) + 6(c^2)^2(\frac{1}{2}x)^2 - 4(c^2)(\frac{1}{2}x)^3 \\ + (\frac{1}{2}x)^4 = c^8 - 2c^6x + \frac{3}{2}c^4x^2 - \frac{1}{2}c^2x^3 + \frac{1}{16}x^4.$$

246. Similarly, a polynomial of three or more terms may be raised to any power by first writing it under the form of a binomial. Thus:

$$(x^2 + 2x + 3)^3 = \{x^2 + (2x + 3)\}^3 \\ = (x^2)^3 + 3(x^2)^2(2x + 3) + 3(x^2)(2x + 3)^2 + (2x + 3)^3 \\ = x^6 + 6x^5 + 21x^4 + 44x^3 + 63x^2 + 54x + 27.$$

Simplify, or expand the following:

13. $(x + 3)^3$. 14. $(x^2 - 2a)^3$. 15. $(a^2 + a + 1)^3$.
 16. $(x - 2)^3$. 17. $(a^2 - 3a)^4$. 18. $(x^2 - x - 1)^2$.
 19. $(a + 4)^4$. 20. $(a^3 + 4c)^3$. 21. $(1 - 2a + a^2)^4$.
 22. $(1 + 2a)^5$. 23. $(2a^2 - 3y)^4$. 24. $(5 - 2x - x^2)^3$.
 25. $(2a - 1)^5$. 26. $(2a^3 - \frac{1}{2}x)^5$. 27. $(x^2 - 5x + 2)^3$.

EVOLUTION.

ROOTS OF MONOMIALS.

247. Since the third power of $2a^2b$ is $2^3a^6b^3$, the cube root of $2^3a^6b^3$ is $2a^2b$.

Since the fourth power of $+3a^3b^2$ or of $-3a^3b^2$ is $+3^4a^{12}b^8$, the fourth root of $+3^4a^{12}b^8$ is $\pm 3a^3b^2$.

Hence, to find the root of a monomial, *divide the exponent of each factor by the index of the root, and take the product of the resulting factors.*

248. Since the third power of $\frac{n^3}{m}$ is $\frac{n^9}{m^3}$, the cube root of $\frac{n^9}{m^3}$ is $\frac{n^3}{m}$.

Hence, to find the root of a fraction, *extract the required root of the terms separately.*

249. It is evident that (242),

1°. *Any even root of a positive number will have the double sign, $\pm$.*

2°. *There can be no even root of a negative number.*

3°. *Any odd root of a number will have the same sign as the number.*

$$\text{Thus, } \sqrt{16a^4} = \pm 4a^2; \quad \sqrt[3]{-8a^6} = -2a^2.$$

But $\sqrt{-a^2}$ is neither $+a$ nor $-a$, for $(+a)^2 = +a^2$, and $(-a)^2 = +a^2$.

250. The indicated even root of a negative number is called an **Imaginary** or **Impossible Number**.

EXERCISE L.

Simplify:

1. $\sqrt{a^4}$.
2. $\sqrt[5]{-32a^{10}b^5}$.
3. $\sqrt{144a^8b^6c^2}$.
4. $\sqrt[3]{a^9}$.
5. $\sqrt{81x^6y^2}$.
6. $\sqrt[3]{216a^3x^9z^6}$.
7. $\sqrt[3]{64x^6}$.
8. $\sqrt[3]{27a^{12}b^3}$.
9. $\sqrt[2]{121a^{10}b^4x^6}$.
10. $\sqrt[4]{16a^8}$.
11. $\sqrt[3]{-125c^6y^{15}}$.
12. $\sqrt[3]{-a^2b^{12}c^9x^3}$.
13. $\sqrt{\frac{4x^4}{9a^2}}$.
14. $\sqrt{\frac{49a^4x^6}{64c^6}}$.
15. $\sqrt[3]{-\frac{8a^6c^9x^{12}}{27b^3y^{12}}}$.
16. $\sqrt{16} + \sqrt{25}$.
17. $\sqrt{25} + \sqrt{16} + \sqrt[3]{-8}$.
18. $\sqrt{81a^2} - \sqrt{49a^2}$.
19. $\sqrt{100a^4b^2} + \sqrt[3]{125a^6b^3}$.
20. $\sqrt[3]{64a^3} + \sqrt{36a^2}$.
21. $\sqrt[4]{16a^8x^{12}} + \sqrt[3]{-27a^6x^9}$.

SQUARE ROOTS OF POLYNOMIALS

251. Since the square root of $a^2 + 2ab + b^2$ is $a + b$, we may determine a general rule for extracting the square root of a polynomial by observing the manner in which $a + b$ may be derived from $a^2 + 2ab + b^2$.

$$\begin{array}{r}
 a^2 + 2ab + b^2(a + b) \\
 a^2 \\
 \hline
 2a + b \left| \begin{array}{l} + 2ab + b^2 \\ + 2ab + b^2 \end{array} \right.
 \end{array}$$

The first term is a^2 , and its square root is a , which is the first term of the root. Subtract its square, a^2 , from the polynomial, and bring

down the remainder $+ 2ab + b^2$. Now we may obtain the second term of the root, $+ b$, by dividing the first term of the remainder, $+ 2ab$, by the double of the part of the root already found, $2a$. Next, we add this new term, b , to twice the part of the root already found, $2a$, for a complete divisor, $2a + b$. Multiply this divisor by the new term, b , and subtract the product from the remainder.

EXERCISE LI.

1. Extract the square root of $25x^4 - 40x^2y + 16y^2$.

$$\begin{array}{r}
 25x^4 - 40x^2y + 16y^2 \quad (5x^2 - 4y) \\
 25x^4 \\
 \hline
 10x^2 - 4y \left| \begin{array}{l} - 40x^2y + 16y^2 \\ - 40x^2y + 16y^2 \end{array} \right.
 \end{array}$$

The square root of the first term is $5x^2$, which is the first term of the root. The second term of the root, $- 4y$, is obtained by dividing $- 40x^2y$ by 2 times $5x^2$, $10x^2$, and this new term is also added to the trial divisor, $10x$, to complete the divisor, $10x^2 - 4y$.

252. The same method will apply to polynomials of more than three terms, if, after each subtraction, the part of the root already found be doubled for a trial

divisor, and the new term of the root be added to the trial divisor for a complete divisor.

Before beginning the work, always arrange the polynomial in reference to a certain letter.

2. Extract the square root of

$$5x^3 - 12x^3 + 6x + 4x^4 + 1.$$

First arrange the polynomial in reference to x , and then proceed thus:

$$\begin{array}{r}
 4x^4 - 12x^3 + 5x^2 + 6x + 1 \quad (2x^2 - 3x - 1 \\
 \underline{4x^4} \\
 4x^2 - 3x \quad \left| \begin{array}{l} -12x^3 + 5x^2 + 6x + 1 \\ -12x^3 + 9x^2 \end{array} \right. \\
 \hline
 4x^2 - 6x - 1 \quad \left| \begin{array}{l} -4x^2 + 6x + 1 \\ -4x^2 + 6x + 1 \end{array} \right.
 \end{array}$$

Extract the square roots of:

3. $9a^2b^4 + 12ab^2 + 4.$
4. $64x^6 - 80x^4y + 25x^2y^2.$
5. $100a^6b^4c^2 + 20a^3b^2c + 1.$
6. $x^4 - 2x^3 + 3x^2 - 2x + 1.$
7. $x^4 - 4x^3 + 8x + 4.$
8. $4x^4 + 12x^3 + 5x^2 - 6x + 1.$
9. $4x^4 - 4x^3 + 5x^2 - 2x + 1.$
10. $4x^4 - 12ax^3 + 25a^2x^2 - 24a^3x + 16a^4.$
11. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4.$
12. $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6.$
13. $x^4 - x^3 + \frac{x^2}{4} + 4x - 2 + \frac{4}{x^2}.$
14. $\frac{a^4}{4} + \frac{a^3}{x} + \frac{a^2}{x^2} - ax - 2 + \frac{x^2}{a^2}.$

SQUARE ROOTS OF ARITHMETICAL NUMBERS.

253. The method of extracting the square roots of polynomials, when considered in connection with the following principle, enables us to devise a rule for extracting the square roots of arithmetical numbers.

254. Principle.—*The square of a number contains twice as many figures as the number, or twice as many less one.*

Thus, the square of any number:

1°. Between 1 and 10 is a number between 1 and 100.

2°. Between 10 and 100 is a number between 100 and 10,000.

3°. Between 100 and 1000 is a number between 10,000 and 1,000,000.

Hence, if a number be pointed off into periods of two figures each, beginning at the right, there will be as many figures in the integral part of the root as there are periods in the number.

EXERCISE LII.

1. Extract the square root of 4225.

Separate the number into 2-figure periods. Let a denote the value of the figure in the tens' place of the root, and b that of the figure in the units' place. Then a must be the greatest multiple of 10 which has its square less than 4200; this is 60. Subtract a^2 , that is 3600, from the given number. Divide the remainder by $2a$, that is 120, and the quotient is b , that is 5, which we add to 120 to obtain the complete divisor, 125.

$$\begin{array}{r}
 42,25 \overline{) 60 + 5} = 65. \\
 \underline{36,00} \\
 625 \\
 \underline{625} \\
 0
 \end{array}$$

If the root consist of three places of figures, let a denote the hundreds and b the tens; then having ob-

tained a and b as before, let the hundreds and tens together be considered as a new value of a , and find a new value of b for the units as before. This is illustrated in the solution of the next example.

2. Extract the square root of 132,496.

After separating the number into 2-figure periods, we find 36, the first two figures of the root, as before.

We now consider 360 as the new value of a , and proceed to find the new value of b , 4, as in the last example.

$$\begin{array}{r} 13, 24, 96 \mid \overset{a}{3}60 + \overset{b}{4} \\ 9 \\ \hline 4 \ 24 \\ 3 \ 96 \\ \hline 28 \ 96 \\ 28 \ 96 \\ \hline \end{array}$$

For the sake of brevity the ciphers are usually omitted. Thus :

$$\begin{array}{r} 42, 25 \mid 65 \\ 36 \\ 125 \overline{) \begin{array}{l} 6 \ 25 \\ 6 \ 25 \end{array}} \end{array}$$

$$\begin{array}{r} 13, 24, 96 \mid 364 \\ 9 \\ 66 \overline{) \begin{array}{l} 4 \ 24 \\ 3 \ 96 \\ \hline 28 \ 96 \\ 28 \ 96 \end{array}} \end{array}$$

NOTES.—1°. If any dividend is less than the divisor, write a 0 in the root and also on the right of the divisor, and annex the next period to the dividend for a new dividend.

2°. If there is a remainder after all the periods have been used, annex periods of decimal 0's, and extend the work to any desired degree of exactness, writing + after the root.

3°. To extract the square root of a mixed number, first reduce it to an improper fraction (if the terms of this fraction are perfect squares), or to a mixed decimal number.

Extract the square roots of:

- | | | |
|-------------|----------------|----------------|
| 3. 9,025. | 4. 6,890,625. | 5. 61.1524. |
| 6. 15,625. | 7. 1,364,224. | 8. 978.4384. |
| 9. 85,264. | 10. 1,971,216. | 11. 78,904 81. |
| 12. 54,756. | 13. 2,446,096. | 14. .00537289. |

Extract the square roots of:

15. $\frac{144}{289}$.

16. $\frac{625}{961}$.

17. $\frac{1,156}{522,729}$.

18. $30\frac{1}{4}$.

19. $1442\frac{1}{4}$.

20. $85,069\frac{4}{9}$.

Simplify:

21. $\sqrt{2}$.

22. $\sqrt{\frac{2}{9}}$.

23. $\sqrt{124\frac{1}{2}}$.

24. $\sqrt[3]{3}$.

25. $\sqrt[3]{\frac{3}{16}}$.

26. $\sqrt[3]{250\frac{2}{3}}$.

27. $\sqrt[5]{5}$.

28. $\sqrt[5]{\frac{5}{36}}$.

29. $\sqrt[3]{101\frac{3}{8}}$.

CUBE ROOTS OF POLYNOMIALS.

255. Since the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a + b$, we may determine a general rule for extracting the cube root of a polynomial by observing the manner in which $a + b$ may be derived from $a^3 + 3a^2b + 3ab^2 + b^3$.

$$\begin{array}{r|l}
 a^3 + 3a^2b + 3ab^2 + b^3 & (a + b \text{ The first term of} \\
 a^3 & \text{the polynomial is} \\
 \hline
 3a^2 + 3ab + b^2 & \left| \begin{array}{l} 3a^2b + 3ab^2 + b^3 \\ 3a^2b + 3ab^2 + b^3 \end{array} \right. \\
 & \text{\textit{a}^3, and its cube} \\
 & \text{\textit{root is a, which} \\
 & \text{\textit{is the first term} \\
 & \text{\textit{of the root. Sub-} \\
 & \text{\textit{tract its cube, a}^3,}
 \end{array}$$

from the polynomial, and bring down the remainder. Divide the first term of the remainder by 3 times the square of the first term of the root, $3a^2$, and the quotient is the second term of the root, b .

Now, take for a complete divisor $3a^2 + 3ab + b^2$, or the trial divisor plus 3 times the product of the first term by the second plus the square of the second, multiply this divisor by the second term, b , and subtract the product from the remainder. In the above, $3a^2$ is called the trial divisor.

EXERCISE LIII.

1. Extract the cube root of $8y^3 + 12xy^2 + 6x^2y + x^3$.

$$\begin{array}{r|l} 8y^3 + 12xy^2 + 6x^2y + x^3 & 2y + x \\ \hline 12y^2 & 8y^3 \\ + 6xy + x^2 & + 12xy^2 + 6x^2y + x^3 \\ \hline 12y^2 + 6xy + x^2 & + 12xy^2 + 6x^2y + x^3 \end{array}$$

The cube root of the first term is $2y$, which is the first term of the root. The second term of the root, $+x$, is obtained by dividing $+12xy^2$ by $3 \times (2y)^2$, or $12y^2$. Now, to the trial divisor $12y^2$, we add $3 \times 2yx + x^2$, and obtain the complete divisor, $12y^2 + 6xy + x^2$

256. The same method may be applied to polynomials of a greater number of terms, provided that, after each subtraction, the part of the root already found be regarded and treated as the first term of the root.

2. Extract the cube root of $64x^6 - 144x^5 + 156x^4 - 99x^3 + 39x^2 - 9x + 1$.

$$\begin{array}{r|l}
 & 4x^2 - 3x + 1 \\
 64x^6 - 144x^5 + 156x^4 - 99x^3 + 39x^2 - 9x + 1 & \\
 \hline
 48x^4 & 64x^6 \\
 \quad - 36x^3 + 9x^2 & - 144x^5 + 156x^4 - 99x^3 + 39x^2 - 9x + 1 \\
 \hline
 48x^4 - 36x^3 + 9x^2 & - 144x^5 + 108x^4 - 27x^3 \\
 \hline
 3(4x^2 - 3x)^2 = & \\
 \quad 48x^4 - 72x^3 + 27x^2 & 48x^4 - 72x^3 + 39x^2 - 9x + 1 \\
 \quad \quad + 12x^2 - 9x + 1 & \\
 \hline
 48x^4 - 72x^3 + 39x^2 - 9x + 1 & 48x^4 - 72x^3 + 39x^2 - 9x + 1
 \end{array}$$

For convenience of arrangement the root is placed above the polynomial.

The first term of the root, $4x^2$, is found by taking the cube root of the first term of the polynomial, and the first trial divisor, $48x^4$, is obtained by taking 3 times the square of the first term of the root.

The first complete divisor is found by adding to the trial divisor $3 \times (4x^2) \times (-3x) + (-3x)^2$.

The part of the root already found is $4x^2 - 3x$, which we now call the first term of the root. Hence, $3(4x^2 - 3x)^2 = 48x^4 - 36x^3 + 27x^2$ is the trial divisor, to which we add $3x(4x^2 - 3x) \times 1 + 1$, or $12x^2 - 9x + 1$, to obtain the complete divisor.

Find the cube roots of:

3. $a^3 - 9a^2 + 27a - 27$.
4. $y^3 + 12y^2 + 48y + 64$.
5. $8a^3 - 36a^2b + 54ab^2 - 27b^3$.
6. $x^6 + 6x^5 - 40x^3 + 96x - 64$.
7. $x^6 - 12x^5 + 54x^4 - 112x^3 + 108x^2 - 48x + 8$.
8. $8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1$.
9. $8x^6 + 48cx^5 + 60c^2x^4 - 80c^3x^3 - 90c^4x^2$
 $+ 108c^5x - 27c^6$.
10. $x^9 + 6x^8 - 64x^6 - 96x^5 + 192x^4 + 512x^3$
 $- 768x - 512$.
11. $a^3 + 3a^2(x + 5) + 3a(x^2 + 10x + 25) + x^3$
 $+ 15x^2 + 75x + 125$.
12. $a^3 - b^3 + c^3 - 3(a^2b - a^2c - ab^2 - ac^2 - b^2c$
 $+ bc^2) - 6abc$.

CUBE ROOTS OF ARITHMETICAL NUMBERS.

257. The method of extracting the cube roots of polynomials, when considered in connection with the following principle, enables us to devise a rule for extracting the cube roots of Arithmetical numbers.

258. Principle. *The cube of a number contains three times as many figures as the number, or three times as many less one or two.*

For, the cube of any number:

- 1°. Between 1 and 10, is a number between 1 and 1000.

2°. Between 10 and 100, is a number between 1,000 and 1,000,000.

3°. Between 100 and 1000, is a number between $(100)^3$ and $(1000)^3$.

Hence, if a number be pointed off into periods of three figures each, beginning at the right, there will be as many figures in the integral part of the root as there are periods in the number.

EXERCISE LIV.

1. Find the cube root of 15,625.

$$\begin{array}{rcl}
 & & 15,625 \mid \overset{a}{20} + \overset{b}{5} \\
 a^3 = & 8,000 & \underline{7,625} \\
 3a^2 = 3 \times (20)^2 = 3 \times 2^2 \times 100 = 1200 & & \\
 3ab = 3 \times 20 \times 5 & = 300 & \\
 b^2 = 5^2 & = 25 & \\
 & 1525 & \underline{7,625}
 \end{array}$$

Separate the number into 3-figure periods.

Let a denote the value of the figure in the tens' place of the root, and b that of the figure in the units' place. Then a must be the greatest multiple of 10, which has its root less than 15,000; this is 20. Subtract a^3 , or 8000, from the given number. Divide the remainder by the trial divisor, $3a^2$, which is 1200, and the quotient is b , that is 5. Next, complete the divisor by adding to the trial divisor $3ab + b^2$, or $300 + 25 = 325$, which gives 1525. Multiplying this by b , or 5, and subtracting the product from 7625, we find the remainder to be 0; hence, $20 + 5$, or 25, is the required cube root.

If the root contains three figures, proceed as in square root. See bottom of page 152.

2. Find the cube root of 12,812,904.

$$\begin{array}{r}
 12,812,904 \mid \overset{a}{230} + \overset{b}{4} \\
 2^3 = 8 \\
 \begin{array}{r}
 3 \times 2^2 \times 100 = 1,200 \\
 3 \times 2 \times 3 \times 10 = 180 \\
 3^2 = 9
 \end{array}
 \begin{array}{r}
 4,812 \\
 180 \\
 9 \\
 \hline
 1,389
 \end{array}
 \begin{array}{r}
 4,167 \\
 645,904 \\
 \hline
 161,476
 \end{array}
 \begin{array}{r}
 158,700 \\
 2,760 \\
 16 \\
 \hline
 161,476
 \end{array}
 \begin{array}{r}
 645,904 \\
 \hline
 645,904
 \end{array}
 \end{array}$$

After separating the number into 3-figure periods, we find 23, the cube root of the greatest cube contained in 12,812, as before. We then consider 230 as the new value of a , and proceed to find the new value of b , 4, as in the last example.

NOTE.—The notes under Art. 254 are applicable to cube root with the change of “square” to “cube.”

Find the cube roots of:

3. 13824.
4. 77308776.
5. 421875.
6. 109.215352.
7. .019683.
8. .726572699.
9. 34.965783.
10. 122615327232.
11. Find the cube root of $\frac{1331}{4913}$; of $\frac{59319}{68921}$.
12. Find the cube root of 2 to three decimal places.
13. Find the cube root of 3 to three decimal places.
14. Find the cube root of 5 to three decimal places.

FRACTIONAL AND NEGATIVE EXPONENTS.

SIGNIFICATION AND PRINCIPLES.

259. *To find the meaning of a fractional exponent.*

The sixth power of a is..... a^6 .

The cube root of the sixth power of a is..... $a^{\frac{6}{3}} = a^2$.

In general, the m th power of a is..... a^m .

The n th root of the m th power of a is..... $a^{\frac{m}{n}}$.

Hence, whether m be divisible by n or not, $a^{\frac{m}{n}}$ may denote $\sqrt[n]{a^m}$, or *the n th root of the m th power of a .*

$$\text{Thus: } a^{\frac{2}{3}} = \sqrt[3]{a^2}; \quad x^{\frac{3}{4}} = \sqrt[4]{x^3}.$$

260. *To find the meaning of a negative exponent.*

By Art. 128. we have $a^0 = 1 \dots (1)$

$$(1) \div a^3 \text{ (127),} \quad a^{-3} = \frac{1}{a^3}.$$

$$\text{In general, } (1) \div a^n, \quad a^{-n} = \frac{1}{a^n}.$$

Hence, a^{-n} is the reciprocal of a^n .

$$\text{Thus, } a^{-2} = \frac{1}{a^2}; \quad \left(\frac{a}{b}\right)^{-2} = \left(\frac{b}{a}\right)^2; \quad \frac{1}{a^{-4}} = a^4.$$

Hence, a factor may be changed from the numerator to the denominator, or vice versa, if the sign of its exponent be changed.

$$\text{Thus, } \frac{a^2 c^{-3}}{b^{-1} n} = \frac{a^2 b}{c^3 n} = a^2 b c^{-3} n^{-1} = \frac{1}{a^{-2} b^{-1} c^3 n}.$$

261. It has been shown that, when m and n are integers,

$$\begin{array}{ll} \text{I. } a^m \times b^m = (ab)^m. & \text{III. } a^m \div a^n = a^{m-n}. \\ \text{II. } a^m \times a^n = a^{m+n}. & \text{IV. } (a^m)^n = a^{mn}. \end{array}$$

It is now to be shown that these formulas are true, whether m and n be integral or fractional, positive or negative.

NOTES.—1. In addition to the notation under Art. 222, we shall employ the following:

(1) ^{n} is read, taking the n th power of both sides of equation 1;

(2) ^{$\frac{1}{n}$} is read, taking the n th root of both sides of equation 2; and similarly for other numbers.

2. For the present, p , q and r will denote only positive numbers; and s and t will represent positive, entire or fractional numbers.

262. Case I.—To prove that Formula I is true:

1°. When m is a fraction.

We have $a \times b = ab \dots\dots (1)$

(1) ^{p} , Art. 240, $a^p \times b^p = (ab)^p \dots\dots (2)$

(2) ^{$\frac{1}{r}$} , Arts. 247, 259, $a^{\frac{p}{r}} \times b^{\frac{p}{r}} = (ab)^{\frac{p}{r}} \dots\dots (3)$

2°. When m is negative.

We have $\frac{1}{a} \times \frac{1}{b} = \frac{1}{ab} \dots\dots (1)$

(1) ^{s} , $\frac{1}{a^s} \times \frac{1}{b^s} = \frac{1}{(ab)^s} \dots\dots (2)$

Or, Art. 260, $a^{-s} \times b^{-s} = (ab)^{-s} \dots\dots (3)$

263. Case II.—To prove that Formula II is true:

1°. When m and n are fractions.

We have $a^p \times a^q = a^{p+q} \dots\dots\dots (1)$

(1) ^{$\frac{1}{r}$} Arts. 247, 259, $a^{\frac{p}{r}} \times a^{\frac{q}{r}} = a^{\frac{p}{r} + \frac{q}{r}} \dots\dots\dots (2)$

2°. When m and n are negative.

We have $\frac{1}{a^s} \times \frac{1}{a^t} = \frac{1}{a^{s+t}}$

Or, Art. 260, $a^{-s} \times a^{-t} = a^{-s-t}$.

Thus: $a^{\frac{2}{3}} \times a^{\frac{1}{3}} = a^{\frac{4}{3}}$; $a^{\frac{2}{3}} \times a^{\frac{1}{2}} = a^{\frac{2}{3} + \frac{1}{2}} = a^{\frac{5}{6}}$.

264. The process of proving the generality of Formula III, is similar to that of Case II.

265. Case IV.—*To prove that Formula IV is true:*

1°. *When n is a fraction.*

We have, Art. 240, $(a^m)^p = a^{m \times p} \dots \dots \dots (1)$

(1) ^{$\frac{1}{r}$} Art. 259, $(a^m)^{\frac{p}{r}} = a^{m \times \frac{p}{r}} \dots \dots \dots (2)$

2°. *When n is negative.*

We have, $\frac{1}{(a^m)^{-s}} = \frac{1}{a^{m \times -s}}$

Or, $(a^m)^{-s} = a^{-ms} = a^{m \times -s}.$

Thus: $(a^3)^{\frac{3}{4}} = a^{\frac{9}{4}}; (a^{\frac{2}{3}})^{-\frac{3}{4}} = a^{-\frac{1}{2}}; (a^{-\frac{3}{4}})^{-\frac{4}{3}} = a.$

EXERCISE LV.

266. Express with fractional exponents:

1. $\sqrt{x}$; $\sqrt[3]{a^2}$; $\sqrt[4]{a^3}$; $(\sqrt[5]{a})^4$; $(\sqrt[3]{a^2})^5$; $\sqrt[6]{a^3b^2}$.
2. $\sqrt[2]{xy^3}$; $\sqrt[4]{a^3b^2c}$; $\sqrt[8]{a^6b^4c^2}$; $3\sqrt[3]{m^2n^4x^3y}$.

Express with radical signs:

3. $a^{\frac{1}{2}}$; $a^{\frac{2}{3}}$; $a^{\frac{1}{2}}b^{\frac{1}{6}}$; $4x^{\frac{1}{3}}y^{\frac{1}{2}}z^{\frac{1}{6}}$; $5x^{\frac{1}{3}}y^{\frac{1}{4}}z^{\frac{1}{6}}$.

Express with positive exponents:

4. a^{-3} ; $2x^{-2}y^{-3}$; $5ab^{-4}$; $x^{-1} + 2x^{-2}y - 3x^{-3}y^2$.

Express as entire numbers:

5. $\frac{a}{b}$; $\frac{ab}{c^2x}$; $\frac{mx^2}{n^4r}$; $\frac{b}{3ac}$; $\frac{c^{\frac{1}{3}}}{4b^{\frac{1}{2}}}$; $\frac{5a^2cx}{b^2dy}$.

Simplify:

6. $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$; $b^{\frac{1}{4}} \times b^{\frac{2}{3}}$; $c^{\frac{3}{4}} \times c^{\frac{5}{6}}$; $x^{\frac{4}{5}} \times x^{\frac{2}{3}}$.
7. $a^{\frac{1}{2}} \times a^{-\frac{1}{3}}$; $n^{\frac{2}{3}} \times n^{-\frac{3}{4}}$; $a^{-2} \times a^{1\frac{1}{2}}$; $m^0 \times m^{\frac{1}{2}}$.

Simplify:

$$8. a^{\frac{1}{2}} \times \sqrt{a}; \sqrt{c} \times c^{-\frac{1}{3}}; \sqrt[3]{c^3} \times \sqrt[3]{c^2}; \sqrt[4]{x^3} \times \sqrt[3]{x^{-4}}$$

$$9. a^{\frac{1}{2}} b^{\frac{2}{3}} c \times ab^{-\frac{1}{2}} c^{\frac{1}{3}}; \quad a^{\frac{3}{4}} b^{\frac{2}{3}} c^{\frac{1}{2}} \times a^{\frac{1}{4}} b^{\frac{1}{6}} c^{-\frac{1}{3}}.$$

$$10. a^{\frac{1}{2}} \times a^{\frac{1}{3}} \times a^{-\frac{1}{4}}; \quad b^{\frac{2}{3}} c^{\frac{1}{4}} \times c^{-\frac{1}{2}} x^{\frac{1}{3}} \times b^{-\frac{1}{2}} x^{-\frac{1}{6}}.$$

$$11. a^{\frac{1}{2}} \div a^{\frac{1}{3}}; \quad c^{\frac{1}{4}} \div c^{\frac{1}{5}}; \quad x^{\frac{3}{5}} \div x^{-\frac{1}{2}}; \quad y^{-\frac{1}{2}} \div y^{\frac{1}{3}}.$$

$$12. (n^4)^{\frac{1}{3}} \div (n^2)^{\frac{1}{5}}; \quad x^{\frac{3}{4}} \div (x^{-2})^{\frac{1}{3}}; \quad 10x^{\frac{1}{2}} y^{\frac{2}{3}} \div 5\sqrt[3]{x} \sqrt{y}.$$

$$13. (m^{\frac{2}{3}})^4; \quad (x^{\frac{3}{4}})^{-2}; \quad (y^{-\frac{1}{2}})^{\frac{2}{3}}; \quad (z^{-\frac{3}{4}})^{-\frac{4}{3}}; \quad (a^{\frac{2}{3}} b^{-\frac{1}{2}})^{-6}.$$

$$14. (2x^2 y^{\frac{2}{3}} z^{-\frac{1}{2}})^{-3}; \quad (-3a^{-2} b^{-\frac{3}{4}} c^{\frac{1}{5}} x^{\frac{1}{3}})^{-\frac{2}{3}}.$$

$$15. (3\sqrt{a} \sqrt[3]{b^2} \sqrt[4]{c^3})^6; \quad (-2\sqrt[3]{a} \sqrt[2]{x^{-3}} \sqrt[5]{y^{\frac{2}{3}}})^{-\frac{3}{4}}.$$

$$16. 16^{\frac{3}{4}}. \quad 16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = (\pm 2)^3 = \pm 8.$$

$$17. \left(\frac{8}{27}\right)^{-\frac{2}{3}} \cdot \left(\frac{8}{27}\right)^{-\frac{2}{3}} = \left(\frac{27}{8}\right)^{\frac{2}{3}} = \left(\frac{\sqrt[3]{27}}{\sqrt[3]{8}}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

$$18. (32)^{\frac{3}{5}}; \quad (64)^{-\frac{1}{2}}; \quad (64)^{-\frac{5}{6}}; \quad (5^{\frac{5}{8}} a^{\frac{2}{3}})^{-\frac{3}{5}}.$$

$$19. \text{Multiply } a^{\frac{3}{4}} - a^{\frac{1}{2}} b^{\frac{1}{4}} + a^{\frac{1}{4}} b^{\frac{1}{2}} - b^{\frac{3}{4}} \text{ by } a^{\frac{1}{4}} + b^{\frac{1}{4}}.$$

$$\begin{array}{r} a^{\frac{3}{4}} - a^{\frac{1}{2}} b^{\frac{1}{4}} + a^{\frac{1}{4}} b^{\frac{1}{2}} - b^{\frac{3}{4}} \\ a^{\frac{1}{4}} + b^{\frac{1}{4}} \\ \hline a - a^{\frac{3}{4}} b^{\frac{1}{4}} + a^{\frac{2}{4}} b^{\frac{1}{2}} - a^{\frac{1}{4}} b^{\frac{3}{4}} \\ + a^{\frac{3}{4}} b^{\frac{1}{4}} - a^{\frac{2}{4}} b^{\frac{1}{2}} + a^{\frac{1}{4}} b^{\frac{3}{4}} - b \\ \hline a \qquad \qquad \qquad - b \end{array}$$

CHAPTER VII.

RADICALS.

DEFINITIONS.

267. A **Radical** is a root of a number, indicated by a radical sign or a fractional exponent.

$$\text{As, } \sqrt[3]{4}, a^{\frac{3}{4}}, \sqrt{a+b}, (a-x)^{\frac{1}{3}}.$$

When the root can be exactly obtained, it is called a *Rational Number*; and when it cannot be exactly obtained, it is called an *Irrational Number* or *Surd*.

268. The **Degree** of a radical is denoted by the index of the radical sign, or by the denominator of the fractional exponent. Thus:

1°, $\sqrt{19}$, or $a^{\frac{1}{2}}$, is a radical of the second degree.

2°, $\sqrt[3]{5}$, or $b^{\frac{1}{3}}$, is a radical of the third degree.

3°, $\sqrt[4]{3}$, or $x^{\frac{2}{4}}$, is a radical of the fourth degree.

269. A **Quadratic Radical** is one of the second degree.

270. If an indicated root is multiplied by a number, the former is called the **Radical Factor**, and the latter the **Coefficient** of the radical factor.

Thus, in $5\sqrt{2}$ and $(a+b)^{\frac{1}{2}}c$, $\sqrt{2}$ and $(a+b)^{\frac{1}{2}}$ are the radical factors, and 5 and c are the coefficients.

271. Similar Radicals are those whose radical factors are the same.

$$\text{As, } 5\sqrt{2} \text{ and } a\sqrt{2}; m\sqrt[3]{b^2c} \text{ and } n\sqrt[3]{b^2c}.$$

272. A surd is in its simplest form when the number under the radical sign is a whole number and as small as possible.

REDUCTION.

EXERCISE LVI.

273. To reduce a rational number to a radical.

1. Reduce 5 to a radical of the 3d degree.

$$5 = 5^{\frac{3}{3}} = (5^3)^{\frac{1}{3}} = \sqrt[3]{125}.$$

2. Reduce $2a^2b$ to a radical of the 4th degree.

$$2a^2b = (2a^2b)^{\frac{4}{4}} = \sqrt[4]{(2a^2b)^4} = \sqrt[4]{16a^8b^4}.$$

Complete the second members of:

$$3. 5a = \sqrt{\quad}. \quad 4. \pm 4a^3b^2c = \sqrt[3]{\quad}.$$

$$5. -2a^2 = \sqrt{\quad}. \quad 6. 2x + 1 = \sqrt{\quad}.$$

$$7. 3c^2x = \sqrt[3]{\quad}. \quad 8. (x-2) = \sqrt[3]{\quad}.$$

$$9. -3ax^2 = \sqrt[4]{\quad}. \quad 10. (x^2 + x)^{\frac{1}{2}} = \sqrt[4]{\quad}.$$

$$11. \frac{2a}{3} = \sqrt[3]{\quad}. \quad 12. -\frac{5a^2c}{6b} = \sqrt[3]{\quad}.$$

274. To carry the coefficient of a radical under the radical sign.

13. In $5a\sqrt{2b}$, carry $5a$ under the radical.

$$5a\sqrt{2b} = \sqrt{(5a)^2} \sqrt{2b} = \sqrt{25a^2 \times 2b} = \sqrt{50a^2b}.$$

14. Complete this: $2x^2\sqrt[3]{5x} = \sqrt[3]{\quad}.$

$$\sqrt[3]{(2x^2)^3} \sqrt[3]{5x} = \sqrt[3]{8x^6 \times 5x} = \sqrt[3]{40x^7}.$$

Complete the following:

15. $6\sqrt{5} = \sqrt{\quad}$. 16. $(a+b)\sqrt{a} = \sqrt{\quad}$.
 17. $3\sqrt[4]{3} = \sqrt[4]{\quad}$. 18. $(x-2)\sqrt{x+2} = \sqrt{\quad}$.
 19. $3a^2b\sqrt{2a} = \sqrt[3]{\quad}$.
 20. $(a-b)\sqrt[3]{c(a-b)^{-3}} = \sqrt[3]{\quad}$.
 21. $-2a^3b^2\sqrt[3]{3a^2b} = \sqrt[3]{\quad}$.
 22. $\frac{a}{c}\sqrt{\frac{c}{a} + \frac{a}{c}} = \sqrt{\quad}$.
 23. $\frac{3}{4}\sqrt[3]{\frac{4}{3}} = \sqrt[3]{\quad}$.
 24. $x\sqrt[3]{\frac{1}{x} - \frac{a}{x^3} + \frac{a^2}{x^5}} = \sqrt[3]{\quad}$.

275. *To remove a factor from under the radical sign to the coefficient.*

The reduction is performed by reversing the process of Art. 274.

25. Complete this: $5\sqrt{12a^2c} = (\quad)\sqrt{3c}$.
 $12a^2c \div 3c = 4a^2$. $\therefore 5\sqrt{12a^2c} = 5\sqrt{4a^2}\sqrt{3c}$
 $= 5 \times 2a\sqrt{3c} = 10a\sqrt{3c}$.

Complete the following:

26. $\sqrt{8} = (\quad)\sqrt{2}$. 27. $2\sqrt[3]{16} = (\quad)\sqrt[3]{2}$.
 28. $\sqrt{27} = (\quad)\sqrt{3}$. 29. $5\sqrt[3]{500} = (\quad)\sqrt[3]{4}$.
 30. $\sqrt{180} = (\quad)\sqrt{5}$. 31. $3\sqrt[4]{80} = (\quad)\sqrt[4]{5}$.
 32. $\sqrt{567} = (\quad)\sqrt{7}$. 33. $4\sqrt[4]{1250} = (\quad)\sqrt[4]{2}$.
 34. $\sqrt{25a^3b} = (\quad)\sqrt{ab}$. 35. $\sqrt{192a^5b^8c^7} = (\quad)\sqrt{3ac}$.
 36. $\sqrt[3]{432a^4b^3} = (\quad)\sqrt[3]{2a}$. 37. $7\sqrt{625a^4b^4c} = (\quad)\sqrt{c}$.
 38. $\frac{1}{2}\sqrt[3]{\frac{8}{7}} = (\quad)\sqrt{21}$. 39. $6\sqrt[3]{\frac{2}{3}} = (\quad)\sqrt[3]{18}$.

Reduce to simplest forms :

40. $3\sqrt{a^4b}$. 41. $5\sqrt{8a^3b^2}$. 42. $2\sqrt{98a^5b^4x^3}$.
 43. $3\sqrt{75x^4y}$. 44. $\sqrt{192a^5b^8c^7}$. 45. $6a\sqrt{180a^3b^5c^2}$.
 46. $\sqrt[3]{81a^4b}$. 47. $\sqrt[3]{160x^5y^3}$. 48. $2\sqrt[3]{108x^6z}$.
 49. $\sqrt[4]{80a^4c}$. 50. $\sqrt[4]{162x^5b^3c^7}$. 51. $4a\sqrt[5]{64a^{10}b^5c}$.
 52. $\sqrt{18x^3y^4 - 27x^4y^3}$. 53. $\sqrt{ax^2 - 6ax + 9a}$.
 54. $\sqrt{(x^2 - y^2)(x + y)}$. 55. $\sqrt[3]{192a^4b^5 + 320a^3b^4}$.
 56. $\sqrt{\frac{2}{3}}$. $\sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3} \times \frac{3}{3}} = \sqrt{\frac{1}{3} \times 6} = \frac{1}{3}\sqrt{6}$.
 57. $\sqrt{\frac{5}{8}}$. 58. $\sqrt{\frac{5}{6}}$. 59. $\sqrt{\frac{7}{18}}$. 60. $\sqrt{\frac{11}{15}}$.
 61. $\frac{1}{2}\sqrt{\frac{3}{7}}$. 62. $\frac{3}{4}\sqrt{\frac{2}{3}}$. 63. $\frac{5}{6}\sqrt{\frac{12}{5}}$. 64. $\frac{a}{b}\sqrt{\frac{b}{a}}$.
 65. $\sqrt[3]{\frac{3}{4}}$. 66. $\sqrt[3]{\frac{2}{25}}$. 67. $\frac{14}{11}\sqrt[3]{\frac{1}{7}}$. 68. $\frac{a}{b}\sqrt[3]{\frac{b}{a}}$.

276. To change the degree of a radical.

69. Complete this: $\sqrt[3]{2a^2c} = \sqrt[12]{\quad}$.
 $12 \div 3 = 4$; $(2a^2c)^4 = 16a^8c^4$. $\therefore \sqrt[3]{2a^2c} = \sqrt[12]{16a^8c^4}$.
 70. Complete this: $\sqrt[6]{8a^4} = \sqrt[2]{\quad}$.
 $2 \div 6 = \frac{1}{3}$; $(8a^4)^{\frac{1}{3}} = 2a^{\frac{4}{3}}$. $\therefore \sqrt[6]{8a^4} = \sqrt[2]{2a^{\frac{4}{3}}}$.

Complete the following:

71. $\sqrt[4]{64a^2b^2} = \sqrt{\quad}$. 72. $\sqrt[2]{3a^2x} = \sqrt[6]{\quad}$.
 73. $\sqrt[3]{16a^4b^4c^{12}} = \sqrt{\quad}$. 74. $\sqrt[3]{2ab^2x^3} = \sqrt[15]{\quad}$.
 75. $\sqrt[6]{25a^2b^2c^4} = \sqrt[3]{\quad}$. 76. $\sqrt[4]{5ax^2} = \sqrt[8]{\quad}$.
 77. $\sqrt[9]{a^6b^9c^3} = \sqrt[3]{\quad}$. 78. $\sqrt[5]{-4a^3z} = \sqrt[10]{\quad}$.

79. $(a^2)^{\frac{1}{4}} = (\quad)^{\frac{1}{2}}$; $(a^3b^2)^{\frac{1}{6}} = (\quad)^{\frac{1}{12}}$; $(a^4b^8)^{\frac{1}{4}} = (\quad)^{\frac{1}{12}}$.
80. $(x^3)^{\frac{1}{6}} = (\quad)^{\frac{1}{2}}$; $(a^3b^6)^{\frac{1}{6}} = (\quad)^{\frac{1}{2}}$; $(a^6b^4)^{\frac{1}{6}} = (\quad)^{\frac{1}{2}}$.

277. To reduce radicals to the same degree.

81. Reduce $\sqrt[4]{2}$ and $\sqrt[6]{3}$ to the same degree.

The L. C. M. of 4 and 6 is 12.

$$\sqrt[4]{2} = \sqrt[12]{2^3} = \sqrt[12]{8}; \quad \sqrt[6]{3} = \sqrt[12]{3^2} = \sqrt[12]{9}.$$

Reduce to a common index:

82. $\sqrt[4]{2}$, $\sqrt[3]{3}$ and $\sqrt[4]{5}$. 83. $3^{\frac{1}{3}}$, $2^{\frac{1}{2}}$, and $5^{\frac{1}{6}}$.
84. $\sqrt[4]{3}$, $\sqrt[3]{2}$ and $\sqrt[6]{7}$. 85. $4^{\frac{3}{4}}$, $2^{\frac{2}{3}}$, and $7^{\frac{1}{2}}$.
86. $\sqrt[6]{3}$, $\sqrt[4]{2}$ and $\sqrt[10]{4}$. 87. $3^{\frac{3}{8}}$, $7^{\frac{5}{6}}$, and $4^{\frac{3}{4}}$.
88. $\sqrt[10]{a^5}$, $\sqrt[6]{b^3}$ and $\sqrt[5]{c^4}$. 89. $\sqrt[7]{c^2}$, $\sqrt[4]{x^3}$ and $\sqrt[14]{z}$.

278. To compare surds.

EXERCISE LVII.

1. Which is the greater, $9\sqrt[4]{2}$ or $7\sqrt[4]{3}$?

$$9\sqrt[4]{2} = \sqrt[4]{81} \times 2 = \sqrt[4]{162}; \quad 7\sqrt[4]{3} = \sqrt[4]{49} \times 3 = \sqrt[4]{147}.$$

$\therefore 9\sqrt[4]{2}$ is greater than $7\sqrt[4]{3}$.

2. Which is the greater, $\sqrt[2]{2}$ or $\sqrt[3]{3}$?

$$\sqrt[2]{2} = \sqrt[6]{8}; \quad \sqrt[3]{3} = \sqrt[6]{9}, \quad \sqrt[3]{3} \text{ is greater than } \sqrt[2]{2}.$$

Which is the greater :

- | | |
|--|---|
| 3. $3\sqrt{7}$ or $2\sqrt{15}$? | 4. $5\sqrt{6}$ or $6\sqrt{5}$? |
| 5. $6\sqrt{11}$ or $5\sqrt{15\frac{1}{2}}$? | 6. $11\sqrt{6}$ or $5\sqrt{29}$? |
| 7. $\sqrt[3]{2}$ or $\sqrt[5]{3}$? | 8. $\sqrt[4]{4}$ or $\sqrt[5]{5}$? |
| 9. $\sqrt{\frac{3}{5}}$ or $\sqrt[3]{\frac{14}{15}}$? | 10. $\frac{2}{3}\sqrt{7}$ or $\frac{3}{5}\sqrt{10}$? |

279. To add and subtract radicals.

EXERCISE LVIII.

1. Find the sum of $a\sqrt{18a^3x}$ and $\sqrt{32a^5x}$.

$$a\sqrt{18a^3x} = a\sqrt{9a^2 \times 2ax} = 3a^2\sqrt{2ax}.$$

$$\sqrt{32a^5x} = \sqrt{16a^4 \times 2ax} = 4a^2\sqrt{2ax}.$$

$$3a^2\sqrt{2ax} + 4a^2\sqrt{2ax} = (3a^2 + 4a^2)\sqrt{2ax} = 7a^2\sqrt{2ax}.$$

Simplify :

- | | |
|--|---|
| 2. $\sqrt{50} + \sqrt{8}.$ | 3. $3\sqrt{12} + 2\sqrt{75} - 4\sqrt{48}.$ |
| 4. $\sqrt{27} - \sqrt{12}.$ | 5. $4\sqrt{18} - 3\sqrt{50} + 5\sqrt{98}.$ |
| 6. $5\sqrt{a^2c} + a\sqrt{9c}.$ | 7. $\sqrt[3]{320} - 2\sqrt[3]{135} + 3\sqrt[3]{40}.$ |
| 8. $\sqrt{32a^3b^2} + \sqrt{18a^3b^2}.$ | 9. $2\sqrt[3]{243a^6c} + 5\sqrt[3]{192a^6c}.$ |
| 10. $\sqrt[3]{24} + \sqrt[3]{54}.$ | 11. $4\sqrt[3]{500x^3y^2} - 3\sqrt[3]{108x^3y^2}$ |
| 12. $\sqrt{363a^5x^6y} - \sqrt{243a^5x^6y} + 2ax^3\sqrt{3ay}.$ | |
| 13. $\sqrt[3]{256a^6x^4} + a\sqrt[3]{108a^3x^4} - 3a^2x\sqrt[3]{32x}.$ | |
| 14. $\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{6}}.$ | 15. $\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{8}}.$ |
| 16. $\frac{1}{2}\sqrt{\frac{3}{8}} + \frac{3}{4}\sqrt{\frac{1}{6}}.$ | 17. $\frac{a}{b}\sqrt{\frac{m}{n}} - \frac{c}{d}\sqrt{\frac{1}{mn}}.$ |
| 18. $\frac{2}{3}\sqrt[3]{\frac{3}{4}} + \frac{3}{4}\sqrt[3]{\frac{2}{3}}.$ | 19. $\sqrt[3]{\frac{27a^5x}{2b}} - a\sqrt[3]{\frac{a^2x}{2b}}.$ |

280. To multiply and divide radicals.**EXERCISE LIX.**1. Multiply $5\sqrt{2a^3b}$ by $3a\sqrt{6a^2}$.

$$5\sqrt{2a^3b} \times 3a\sqrt{6a^2} = 5 \times 3a\sqrt{2a^3b} \times \sqrt{6a^2} = 15a\sqrt{12a^5b} \\ = 30a^3\sqrt{3ab}.$$

2. Divide $5\sqrt[3]{4}$ by $3\sqrt{2}$.

$$5\sqrt[3]{4} = 5\sqrt[6]{16}; \quad 3\sqrt{2} = 3\sqrt[6]{8}; \quad 5\sqrt[6]{16} \div 3\sqrt[6]{8} = \\ (5 \div 3) \sqrt[6]{16 \div 8} = 1\frac{2}{3}\sqrt[6]{2}.$$

Simplify:

3. $4\sqrt{6a} \times 3\sqrt{8ac}.$

4. $15\sqrt{6a^3x} \div 5\sqrt{2a}.$

5. $7\sqrt{10a^2} \times 2\sqrt{5ab^2}.$

6. $9x\sqrt{54a^2} \div 3x\sqrt{3}.$

7. $5\sqrt{14a^5} \times 2x\sqrt{7a^2c}.$

8. $12a^3\sqrt{96a^3x} \div 4\sqrt{2ax}.$

9. $6\sqrt{12} \times 3\sqrt{3}.$

10. $6\sqrt{20} \div 3\sqrt{15}.$

11. $\frac{1}{4}\sqrt{6} \times \frac{3}{14}\sqrt{8}.$

12. $\frac{2}{3}\sqrt{\frac{3}{4}} \div \frac{1}{2}\sqrt{\frac{4}{3}}.$

13. $4\sqrt[3]{\frac{2}{3}} \times 3\sqrt[3]{\frac{5}{6}}.$

14. $\frac{1}{2}\sqrt[3]{\frac{2}{3}} \div \frac{1}{3}\sqrt[3]{\frac{1}{2}}.$

15. $3\sqrt[3]{4} \times 4\sqrt{3}.$

16. $2\sqrt[3]{3} \div 3\sqrt{2}.$

17. $\sqrt{\frac{1}{2}} \times \sqrt[3]{\frac{2}{3}} \div \sqrt[4]{\frac{1}{3}}.$

18. $\sqrt{\frac{3}{4}} \times \sqrt[5]{\frac{2}{3}} \div \sqrt[10]{\frac{4}{3}}.$

NOTE.—The n th power of the n th root of a number is equal to that number.

$$\text{Thus, } (\sqrt{7})^2 = 7, \quad (\sqrt[3]{1+x+x^2})^3 = 1+x+x^2.$$

19. Multiply $\sqrt{2} + 4\sqrt{3}$ by $\sqrt{2} - 7\sqrt{3}$.

$$\begin{array}{r} \sqrt{2} + 4\sqrt{3} \\ \sqrt{2} - 7\sqrt{3} \\ \hline \end{array}$$

$$\begin{array}{r} 2 + 4\sqrt{6} \\ - 7\sqrt{6} - 28 \times 3 \\ \hline \end{array}$$

$$\begin{array}{r} - 7\sqrt{6} - 28 \times 3 \\ \hline \end{array}$$

$$2 - 3\sqrt{6} - 84 = -82 - 3\sqrt{6}.$$

Multiply:

20. $\sqrt{7} + \sqrt{5}$ by $\sqrt{7} - \sqrt{5}$.

21. $\sqrt{6} + \sqrt{8}$ by $\sqrt{6} + \sqrt{8}$.

22. $2\sqrt{3} - 5\sqrt{2}$ by $3\sqrt{3} + 4\sqrt{2}$.

23. $5 + 3\sqrt{7}$ by $9 - 4\sqrt{7}$.

24. $\sqrt{2} + \sqrt{3} + \sqrt{5}$ by $\sqrt{2} - \sqrt{3} + \sqrt{5}$.

25. $3\sqrt{5} - 2\sqrt{6} + \sqrt{7}$ by $6\sqrt{5} + 4\sqrt{6} + 2\sqrt{7}$.

26. $\sqrt{a} + \sqrt{b} + \sqrt{c}$ by $\sqrt{a} - \sqrt{b} + \sqrt{c}$.

27. $\sqrt{1+x} + 2\sqrt{1-x}$ by $\sqrt{1+x} - 5\sqrt{1-x}$.

28. $\sqrt{2+3x} + 7\sqrt{3+2x}$ by $3\sqrt{2+3x} - 4\sqrt{3+2x}$.

29. $\sqrt{1+x} + 2\sqrt{1+2x} - 3\sqrt{1-3x}$ by $\sqrt{1+x}$
 $- 2\sqrt{1+2x} + 3\sqrt{1-3x}$.

INVOLUTION OF RADICALS.

281.—EXERCISE LX

1. Raise $\sqrt[3]{2}$ to the 9th power.

$$(\sqrt[3]{2})^9 = (2^{\frac{1}{3}})^9 = 2^{\frac{9}{3}} = 2^3 = 8.$$

2. Raise $2\sqrt[2]{3}$ to the 5th power.

$$(2\sqrt[2]{3})^5 = 2^5 (3)^{\frac{5}{2}} = 2^5 (3^5)^{\frac{1}{2}} = 32\sqrt{243}.$$

Find the:

3. 4th power of $2\sqrt{a}$.

4. 3d power of $5a\sqrt[3]{a}$.

5. 2d power of $3\sqrt[4]{m}$.

6. 4th power of $2x^2\sqrt[3]{6}$.

7. 3d power of $-\sqrt[6]{a^2x}$.

8. 9th power of $-x\sqrt[6]{a^5}$.

9. 6th power of $\frac{2}{3}\sqrt[3]{3}$.

10. 10th power of $-\frac{1}{2}\sqrt[2]{2}$.

11. 2d power of $x\sqrt[2n]{y}$.

12. n th power of $\sqrt[mn]{y}$.

Simplify (245):

13. $(1 + \sqrt{2})^2$. 14. $(1 - \sqrt{3})^3$. 15. $(2 + \sqrt{5})^3$.
 16. $(\sqrt{2} + 3)^2$. 17. $(\sqrt{5} - 1)^4$. 18. $(\sqrt{3} + \sqrt{2})^3$.
 19. $(2 + \sqrt[3]{3})^2$. 20. $(3 - \sqrt[3]{2})^3$. 21. $(\sqrt{2} + \sqrt[3]{4})^3$.
 22. $(a^{\frac{1}{2}} + b^{\frac{1}{3}})^2$. 23. $(a^{\frac{2}{3}} + a^{\frac{1}{6}})^3$. 24. $(\sqrt{a} + \sqrt[3]{b})^4$.
 25. $(1 + x + \sqrt{1+x})^2$. 26. $(\sqrt{1+x} + \sqrt{1-x})^2$.

EVOLUTION OF RADICALS.

282. EXERCISE LXI.

1. Extract the cube root of $2\sqrt{6}$

$$(2\sqrt{6})^{\frac{1}{3}} = 2^{\frac{1}{3}}(6^{\frac{1}{2}})^{\frac{1}{3}} = 2^{\frac{1}{3}}6^{\frac{1}{6}} = \sqrt[3]{2} \sqrt[6]{6}.$$

2. Extract the fourth root of $a^4\sqrt[3]{16}$

$$(a^4\sqrt[3]{16})^{\frac{1}{4}} = (a^4)^{\frac{1}{4}} (16^{\frac{1}{3}})^{\frac{1}{4}} = a(16^{\frac{1}{4}})^{\frac{1}{3}} = a(2)^{\frac{1}{3}} = a\sqrt[3]{2}.$$

Find the:

3. Cube root of $8\sqrt{a^3}$. 4. Square root of $5\sqrt[3]{16}$.
 5. Fourth root of $9\sqrt[3]{a^2}$. 6. Cube root of $27\sqrt[4]{8}$.
 7. Sixth root of $x^{12}\sqrt{a^3}$. 8. Tenth root of $a^5\sqrt[4]{b^{20}}$.
 9. Cube root of $\frac{a}{3} \sqrt{\frac{a}{3}}$. 10. Fourth root of $\frac{4}{9} \sqrt[3]{\frac{4}{9}}$.
 11. Square root of $x + 2\sqrt{xy} + y$.
 12. Square root of $\sqrt{a} - 2\sqrt[4]{ab} + \sqrt{b}$.
 13. Square root of $\sqrt[3]{a} + 2\sqrt[6]{ab} + \sqrt[3]{b}$.
 14. Square root of $4\sqrt[3]{a^2} - 20\sqrt[3]{ab} + 25\sqrt[3]{b^2}$.
 15. Cube root of $a - 3\sqrt[3]{a^2x} + 3\sqrt[3]{ax^2} - x$.
 16. Square root of $a + 2\sqrt{ab} + b - 2\sqrt{ac} - 2\sqrt{bc} + c$.

283. *To extract the square root of binomial surds.*

17. Extract the square root of $8 + 4\sqrt{3}$.

Problems of this kind may^{*} often be solved by inspection, thus: since the square of $\sqrt{a} + \sqrt{b} = (a+b) + 2\sqrt{ab}$, the square root of $(a+b) + 2\sqrt{ab}$ is $\sqrt{a} + \sqrt{b}$. Hence, if the given expression be written in the form $m + 2\sqrt{n}$, we have only to find two numbers whose sum is m and whose product is n , and connect their square roots by the intermediate sign of the given expression.

$$\text{Thus, } 8 + 4\sqrt{3} = 8 + 2 \times 2\sqrt{3} = 8 + 2\sqrt{12}.$$

Now, we find by inspection that $6 + 2 = 8$ and $6 \times 2 = 12$. Hence, the required root is $\sqrt{6} + \sqrt{2}$.

18. Find the square root of $12 - 8\sqrt{2}$.

$$12 - 8\sqrt{2} = 12 - 2\sqrt{32}. \quad 8 + 4 = 12 \text{ and } 8 \times 4 = 32.$$

$$\text{Hence, } \sqrt{12 - 8\sqrt{2}} = \sqrt{4} - \sqrt{8} = 2 - 2\sqrt{2}.$$

Find the square root of:

$$19. \quad 4 + 2\sqrt{3}. \quad 20. \quad 7 - 4\sqrt{3}. \quad 21. \quad 7 + 2\sqrt{10}.$$

$$22. \quad 18 + 8\sqrt{5}. \quad 23. \quad 17 + 4\sqrt{15}. \quad 24. \quad 13 - 2\sqrt{30}.$$

$$25. \quad 10 + 2\sqrt{21}. \quad 26. \quad 11 - 6\sqrt{2}. \quad 27. \quad 38 - 12\sqrt{10}.$$

$$28. \quad 16 - 2\sqrt{55}. \quad 29. \quad 35 + 10\sqrt{10}. \quad 30. \quad 24 - \sqrt{252}.$$

284. General Solution. Let us suppose

$$\sqrt{m + \sqrt{n}} = \sqrt{a} + \sqrt{b}.$$

Then, from what precedes,

$$a + b = m \dots (1)$$

and

$$4ab = n \dots (2)$$

Square (1),

$$a^2 + 2ab + b^2 = m^2 \dots (3)$$

Subtract (2) from (3),

$$a^2 - 2ab + b^2 = m^2 - n$$

Extract the square root

$$a - b = \sqrt{m^2 - n} \dots (4)$$

Add (1) and (4), reduce,

$$a = \frac{1}{2}(m + \sqrt{m^2 - n}).$$

Subtract (4) from (1),

$$b = \frac{1}{2}(m - \sqrt{m^2 - n}).$$

$$\therefore \sqrt{m + \sqrt{n}} = \sqrt{\frac{1}{2}(m + \sqrt{m^2 - n})} + \sqrt{\frac{1}{2}(m - \sqrt{m^2 - n})}.$$

31. Find the square root of $17 + 4\sqrt{15}$.

$17 + 4\sqrt{15} = 17 + \sqrt{240}$. Here, $m = 17$ and $n = 240$. Substituting in the above we have

$$\begin{aligned} \sqrt{17 + \sqrt{240}} &= \sqrt{\frac{1}{2}(17 + \sqrt{49})} + \sqrt{\frac{1}{2}(17 - \sqrt{49})} \\ &= \sqrt{12} + \sqrt{5}, \text{ Ans.} \end{aligned}$$

The preceding examples may be solved in a similar manner.

IMAGINARY NUMBERS.

285. An **Imaginary Number** is an indicated even root of a negative quantity (**250**); as $\sqrt{-a}$, $\sqrt[4]{-n^2}$.

In contradistinction, all other quantities, rational or irrational, are called *real* quantities.

286. Imaginary numbers may be reduced to one form.

$$\begin{aligned} \text{Thus, } \sqrt{-m^2} &= \sqrt{m^2(-1)} = m\sqrt{-1}, \\ \sqrt{-n} &= \sqrt{n(-1)} = \sqrt{n}\sqrt{-1}. \end{aligned}$$

287. Since $\sqrt{-1}$ is the square root of -1 , $\sqrt{-1} \times \sqrt{-1} = -1$. Therefore,

$$(\sqrt{-1})^2 = -1;$$

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = -\sqrt{-1};$$

$$(\sqrt{-1})^4 = (\sqrt{-1})^3 \times \sqrt{-1} = -(-1) = +1;$$

and so on. Hence the successive powers of $\sqrt{-1}$ form the repeating series, $+\sqrt{-1}, -1, -\sqrt{-1}, +1$.

EXERCISE LXII.

1. Multiply $2 + \sqrt{-9}$ by $2 - \sqrt{-9}$.

$$2 + \sqrt{-9} = 2 + 3\sqrt{-1}, \text{ and } 2 - \sqrt{-9} = 2 - 3\sqrt{-1}$$

$$(2 + 3\sqrt{-1})(2 - 3\sqrt{-1}) = 4 - 9(-1) = 13, \text{ Ans.}$$

2. Divide $\sqrt{-185}$ by $5\sqrt{-1}$.

$$\sqrt{-185} = \sqrt{185} \sqrt{-1}, \text{ and } 5\sqrt{-1} = \sqrt{25} \sqrt{-1}$$

$$\frac{\sqrt{185}\sqrt{-1}}{\sqrt{25}\sqrt{-1}} = \sqrt{\frac{185}{25}} = \sqrt{\frac{37}{5}}, \text{ or } \frac{1}{5}\sqrt{185}.$$

Simplify:

3. $\sqrt{-3} \times \sqrt{-6}.$

4. $\sqrt{-12} \div \sqrt{-3}.$

5. $\sqrt{12} \times \sqrt{-8}.$

6. $\sqrt{15} \div \sqrt{-5}.$

7. $4\sqrt{-3} \times 2\sqrt{-3}.$

8. $8\sqrt{-5} \div 2\sqrt{-20}.$

9. $(2\sqrt{-4})^3.$

10. $(5\sqrt{-3})^2.$

11. $(3\sqrt{-2})^4.$

12. $(2\sqrt{-2})^5.$

13. $(2 + 5\sqrt{-1}) \times (2 - 5\sqrt{-1}).$

14. $(3 + 7\sqrt{-2}) \times (3 - 7\sqrt{-2}).$

15. $(\sqrt{3} + 4\sqrt{-5}) \times (\sqrt{3} - 4\sqrt{-5}).$

$$16. (5\sqrt{-3} + 4\sqrt{-2}) \times (5\sqrt{-3} - 4\sqrt{-2}).$$

$$17. (1 + 2\sqrt{-1})^3.$$

$$18. (1 - 3\sqrt{-1})^2.$$

Verify the following:

$$19. \left(\frac{-1 - \sqrt{-3}}{2} \right)^3 = 1.$$

$$20. \left(\frac{1 - \sqrt{-3}}{2} \right)^3 = -1.$$

$$21. \left(\frac{-1 + \sqrt{-3}}{2} \right)^3 = 1.$$

$$22. \left(\frac{1 + \sqrt{-3}}{2} \right)^3 = -1.$$

SIMPLIFICATION OF IRRATIONAL FRACTIONS.

288. To simplify a fraction having an irrational denominator is to reduce it to an equivalent fraction whose denominator is rational.

The process consists in multiplying both terms by such a number as shall produce the desired result

EXERCISE LXIII.

289. *When the denominator is a monomial.*

Reduce the following to equivalent fractions having rational denominators:

$$1. \frac{2 + x}{\sqrt[3]{5}}.$$

Multiply both terms by $\sqrt[3]{25}$,

$$\frac{(2 + x) \sqrt[3]{25}}{\sqrt[3]{5} \times \sqrt[3]{25}} = \frac{(2 + x) \sqrt[3]{25}}{\sqrt[3]{125}} = \frac{(2 + x) \sqrt[3]{25}}{5}.$$

$$2. \frac{3}{\sqrt{2}}.$$

$$3. \frac{5}{\sqrt{8}}.$$

$$4. \frac{6}{\sqrt{18}}.$$

$$5. \frac{2 + \sqrt{3}}{\sqrt{3}}.$$

$$6. \frac{\sqrt[3]{4} - \sqrt[3]{5}}{2\sqrt[3]{4}}.$$

$$7. \frac{3 + \sqrt{5}}{\sqrt{-2}}.$$

$$8. \frac{1 + \sqrt[4]{5}}{\sqrt[4]{8}}. \quad 9. \frac{6 + 3\sqrt[3]{2}}{\sqrt[3]{9}}. \quad 10. \frac{na + a\sqrt{-2}}{\sqrt[5]{a^4}}.$$

290. *When the denominator is a binomial, containing only radicals of the second degree.*

Find equivalent fractions with rational denominators, for the following:

$$11. \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}}.$$

Multiply both terms by $\sqrt{5} + \sqrt{2}$ (the denominator with its intermediate sign changed),

$$\frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}} = \frac{(\sqrt{5} + \sqrt{2})(\sqrt{5} + \sqrt{2})}{(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})} = \frac{7 + 2\sqrt{10}}{3}.$$

$$12. \frac{4 - \sqrt{3}}{4 + \sqrt{3}}. \quad 13. \frac{3 + \sqrt{5}}{3 - \sqrt{5}}. \quad 14. \frac{\sqrt{2} + \sqrt{3}}{\sqrt{2} - \sqrt{3}}.$$

$$15. \frac{5 + \sqrt{-7}}{5 - \sqrt{-7}}. \quad 16. \frac{\sqrt{-8} + 4}{\sqrt{-8} - 4}. \quad 17. \frac{\sqrt{6} - 3\sqrt{-3}}{\sqrt{6} + 3\sqrt{-3}}.$$

$$18. \frac{2 + \sqrt{a+1}}{2 - \sqrt{a+1}}. \quad 19. \frac{a - \sqrt{a^2-1}}{a + \sqrt{a^2-1}}. \quad 20. \frac{x + \sqrt{x^2-4}}{x - \sqrt{x^2-4}}.$$

$$21. \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}}. \quad 22. \frac{\sqrt{a^2-1} - \sqrt{a^2+1}}{\sqrt{a^2-1} + \sqrt{a^2+1}}.$$

PROPERTIES OF IRRATIONAL NUMBERS.

291. I.—*No irrational number can be expressed by a fraction.*

For, if possible, let
$$\sqrt[n]{a} = \frac{b}{c} \dots \dots (1)$$

Then, (1)ⁿ,
$$a = \frac{b^n}{c^n} \dots \dots (2)$$

Now, if $\frac{b}{c}$ is in its lowest terms, $\frac{b^n}{c^n}$ is in its lowest terms, and

we shall have an integer (a) equal to an irreducible fraction, which is impossible. Hence, (1) can not be true.

292. II.—*A simple quadratic surd can not be equal to the sum of a rational number and a surd.*

For, if possible, let $\sqrt{a} = b + \sqrt{c} \dots \dots (1)$
in which $\sqrt{a}$ and $\sqrt{c}$ are surds.

Square both sides of (1), $a = b^2 + 2b\sqrt{c} + c \dots (2)$

Transpose and reduce, $\sqrt{c} = \frac{a - b^2 - c}{2b} \dots (3),$

that is, $\sqrt{c}$ is equal to a rational fraction, which is impossible. Hence, (1) can not be true.

293. III.—*The product or quotient of two dissimilar quadratic surds is a quadratic surd.*

For, if possible, let $\sqrt{a} \times \sqrt{c} = y \dots \dots (1)$
in which $\sqrt{a}$ and $\sqrt{c}$ are dissimilar surds.

Square both sides, $ac = y^2 \dots \dots (2)$

Whence, $a = \frac{y^2}{c} \dots \dots (3)$

That is, $\frac{y^2}{c}$ is equal to an integer; therefore y is some multiple of c . Hence, in (3) we may make $y = mc$, which gives $a = m^2c$ or $\sqrt{a} = m\sqrt{c}$, which is contrary to the hypothesis that $\sqrt{a}$ and $\sqrt{c}$ are dissimilar surds. Hence, (1) can not be true.

Again, if possible, let $\sqrt{a} \div \sqrt{c} = y \dots \dots (1)$

Square both sides, $\frac{a}{c} = y^2$

Reduce, $a = y^2c$

Or, $\sqrt{a} = y\sqrt{c}$

which is contrary to the hypothesis. Hence, (1) cannot be true.

294. IV.—*The sum or difference of two dissimilar quadratic surds can not be a rational number, nor can it be expressed as a single surd.*

Let $\sqrt{a}$ and $\sqrt{c}$ be two such surds; then, if possible, suppose

$$\sqrt{a} \pm \sqrt{c} = y \dots \dots (1)$$

Square both sides, $a \pm 2\sqrt{ac} + c = y^2$

Whence, $\sqrt{ac} = \frac{y^2 - a - c}{2}.$

But $\sqrt{ac}$ is irrational (293); hence we have an irrational number equal to a rational fraction, which is impossible. Hence, (1) can not be true.

In like manner, it may be shown that $\sqrt{a} \pm \sqrt{c}$ can not be equal to a surd.

295. V.—*In an equation involving rational numbers and simple quadratic surds, the rational numbers of the two members are equal, and also the surds.*

Suppose $x + \sqrt{y} = a + \sqrt{c} \dots \dots \dots (1)$

in which $\sqrt{y}$ and $\sqrt{c}$ are irrational; then will $x = a$ and $y = c$.

For, by transposing, $\sqrt{y} - \sqrt{c} = a - x$, and if c were not equal to y , the difference of two unequal surds would be rational, which is impossible.

$$\therefore c = y \text{ and } a = x.$$

EXERCISE LXIV.

1. Find the rational values of x and y in

$$x + 3 + (y - 8)\sqrt{2} = 8 + (6 - y)\sqrt{2}.$$

By Art. 295 we have

$$x + 3 = 8 \text{ and } (y - 8)\sqrt{2} = + (6 - y)\sqrt{2}$$

Whence, $x = 5$ and $y = 7$.

Find the rational values of x and y in :

2. $x + 4\sqrt{3} = 6 - x + y\sqrt{3}.$
3. $x - 4 + (y - 6)\sqrt{5} = 3x - 13.$
4. $5x - 6 + (9y - 16)\sqrt{2} = 3x + 2 + 5y\sqrt{2}.$
5. $\frac{4x}{9} - 10 + \left(\frac{3y - 2}{4}\right)\sqrt{7} = \left(\frac{27 + 2y}{6}\right)\sqrt{7} + \frac{x}{3} - x.$
6. $3x + 5y + (4x - 8)\sqrt{11} = 2y\sqrt{11} + 19.$
7. $(4x + 6y)\sqrt{2} + 34 = 7x - 3y + 58\sqrt{2}.$

EQUATIONS CONTAINING RADICALS.

296. EXERCISE LXV.

1. Find the value of x in $\sqrt{x^2 - 15} + x - 3 = 0.$

$$\text{Transpose,} \quad \sqrt{x^2 - 15} = -x + 3$$

$$\text{Square,} \quad x^2 - 15 = x^2 - 6x + 9$$

$$\text{Whence,} \quad x = 4$$

2. Find the value of x in $\sqrt{x + 5} + \sqrt{x} = 5.$

$$\text{Transpose,} \quad \sqrt{x + 5} = 5 - \sqrt{x}$$

$$\text{Square,} \quad x + 5 = 25 - 10\sqrt{x} + x$$

$$\text{Transpose, reduce, } 10\sqrt{x} = 20$$

$$\text{Or,} \quad \sqrt{x} = 2$$

$$\text{Square,} \quad x = 4.$$

3. Find the value of x in $\sqrt{x + 1} + \sqrt{x - 2} - \sqrt{4x - 3} = 0.$

$$\text{Transpose,} \quad \sqrt{x+1} + \sqrt{x-2} = \sqrt{4x-3}$$

$$\text{Square,} \quad x+1 + 2\sqrt{x^2-x-2} + x-2 = 4x-3$$

$$\text{Transpose,} \quad 2\sqrt{x^2-x-2} = 2x-2$$

$$\text{Divide by 2,} \quad \sqrt{x^2-x-2} = x-1$$

$$\text{Square,} \quad x^2-x-2 = x^2-2x+1$$

$$\text{Whence,} \quad x = 3.$$

Find the value of x in:

$$4^{\circ} \sqrt{x^2+16} = x+2. \quad 5. \sqrt{x^2+11} = x+1.$$

$$6. \sqrt{4x^2-19} - 2x + 1 = 0.$$

$$7. \sqrt{9x^2-3x+7} - 3x = 5.$$

$$8. \sqrt{x+7} + \sqrt{x} = 7.$$

$$9. \sqrt{x} + \sqrt{x+5} = 5.$$

$$10. 6 + \sqrt{x} = \sqrt{12+x}.$$

$$11. \sqrt{x-3} + \sqrt{x-12} = -3.$$

$$12. \sqrt{x-32} + \sqrt{x} = 16.$$

$$13. \sqrt{3x+1} + \sqrt{3x-6} = 7.$$

$$14. \sqrt{5-x} = \sqrt[4]{5+x^2}.$$

$$15. \sqrt{5x+16} = 10 - \sqrt{5x-4}.$$

$$16. \sqrt[4]{x-16} = \sqrt{8-\sqrt{x}}.$$

$$20. \quad x - \sqrt{x^2 - 3x} = 2.$$

$$21. \quad \sqrt{2x - 3} - \sqrt{8x + 1} + \sqrt{18x - 92} = 0.$$

$$22. \quad \sqrt{c + x} = \frac{\sqrt{a + x^2}}{\sqrt{c + x}}.$$

$$23. \quad \frac{a}{\sqrt{x}} + \frac{\sqrt{x}}{c} = \sqrt{\frac{n}{x}}.$$

$$24. \quad \frac{\sqrt{x} - 3}{\sqrt{x} + 7} = \frac{\sqrt{x} - 4}{\sqrt{x} + 1}.$$

$$25. \quad \frac{\sqrt{x} + 6}{\sqrt{x} + 5} = \frac{\sqrt{x} + 12}{\sqrt{x} + 15}.$$

$$26. \quad \frac{\sqrt{6x} - 2}{\sqrt{6x} + 2} = \frac{4\sqrt{6x} - 9}{4\sqrt{6x} + 6}.$$

$$27. \quad \frac{\sqrt{ax} + b}{\sqrt{ax} - b} = \frac{3\sqrt{ax} + 5b}{3\sqrt{ax} - 2b}.$$

$$28. \quad \frac{\sqrt{3x} - \sqrt{3}}{\sqrt{2x} - \sqrt{2}} = \frac{\sqrt{x} + 3}{\sqrt{x} + 2}.$$

$$29. \quad \frac{\sqrt{4x + 1} + \sqrt{4x}}{\sqrt{4x + 1} - \sqrt{4x}} = 9.$$

$$30. \quad \sqrt{x + a} = \sqrt[4]{a^2 + x\sqrt{b^2 + x^2}}.$$

CHAPTER VIII.

QUADRATIC EQUATIONS.

297. A Quadratic Equation, or an *Equation of the Second Degree*, is one in which the square is the highest power of the unknown number; as,

$$3x^2 = 12, \text{ and } x^2 + 7x = -10.$$

PURE QUADRATIC EQUATIONS.

298. A Pure Quadratic Equation, or an *Incomplete Equation of the Second Degree*, is one which contains only the square of the unknown number; as,

$$x^2 = 25, \text{ and } 3x^2 + 2 = 11 + 2x^2.$$

EXERCISE LXVI.

Find the values of x in

$$3x^2 + 5 = \frac{5x^2}{4} + 33 \dots\dots (1)$$

$$(1) \times 4, \quad 12x^2 + 20 = 5x^2 + 132 \dots\dots (2)$$

$$\text{Transpose, reduce,} \quad 7x^2 = 112 \dots\dots\dots (3)$$

$$(3) \div 7 \quad x^2 = 16 \dots\dots\dots (4)$$

$$\sqrt{(4)} \quad x = \pm 4, \text{ Ans.}$$

Rule. Find the value of x^2 and take its square root.

Find the values of x in $3x^2 + 12 = 0$.

$$3x^2 = -12$$

It will be observed that the roots $x^2 = -4$

in this case are imaginary.

$$x = \pm \sqrt{-4} = \pm 2\sqrt{-1}.$$

Solve :

$$3. \quad x^2 - 5 = 59.$$

$$4. \quad x^2 + 7 = 56.$$

$$5. \quad x^2 - 8 = 2x^2 - 44. \quad 6. \quad 5x^2 - 9 = 2x^2 + 24.$$

$$7. \quad (x + 3)^2 = 6x + 25. \quad 8. \quad (x - 5)^2 = 106 - 10x.$$

$$9. \quad \frac{3}{4x^2} - \frac{1}{6x^2} = \frac{7}{3}. \quad 10. \quad 8x + \frac{7}{x} = \frac{65x}{7}.$$

$$11. \quad \frac{5}{4+x} = \frac{8}{3} - \frac{5}{4-x}. \quad 12. \quad \frac{3}{1+x} + \frac{3}{1-x} = 8.$$

$$13. \quad 12 - \sqrt{3x^2 + 16} = 5. \quad 14. \quad \sqrt{6x - x^2} - 3 = x.$$

$$15. \quad x + \sqrt{x^2 + 3} = \frac{6}{\sqrt{x^2 + 3}}.$$

$$16. \quad \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2} - x} = \frac{1}{4}.$$

$$17. \quad x + \sqrt{a^2 + x^2} = \frac{25a^2}{\sqrt{a^2 + x^2}}.$$

$$18. \quad \frac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2} - x} = \frac{b^2}{c^2}.$$

AFFECTED QUADRATIC EQUATIONS.

299. An Affected Quadratic Equation, or a *Complete Equation of the Second Degree*, is, one which contains both the square and first power of the unknown number ; as

$$x^2 + 5x + 6 = 0, \text{ and } 3x - 2x^2 = 7 - 5x.$$

300. SOLUTION BY FACTORING.

EXERCISE LXVII.

1. Find the values of x in $x^2 + 8x + 15 = 0$.

Art. 152, $x^2 + 8x + 15 = (x + 3)(x + 5) = 0$.

This equation can be satisfied in two ways:

1°. By making $x + 3 = 0$, whence, $x = -3$,

2°. By making $x + 5 = 0$, whence, $x = -5$.

2. Find the values of x in $2x^2 - 28x' = 102$.

Dividing by 2 and transposing, $x^2 - 14x - 51 = 0$.

Factoring, $(x - 17)(x + 3) = 0$.

Let $x - 17 = 0$, $\therefore x = +17$; let $x + 3 = 0$, $\therefore x = -3$.

3. Find the values of x in $12x^2 + 17x - 40 = 0$.

Art. 154, $12x^2 + 17x - 40 = (3x + 8)(4x - 5)$

$3x + 8 = 0$, $\therefore x = -2\frac{2}{3}$; $4x - 5 = 0$, $\therefore x = +1\frac{1}{4}$.

Solve:

- | | |
|----------------------------|---------------------------|
| 4. $x^2 + 11x + 18 = 0$. | 5. $x^2 + 10x + 21 = 0$. |
| 6. $y^2 - 15y + 36 = 0$. | 7. $y^2 - 13y + 22 = 0$. |
| 8. $z^2 + 5z - 24 = 0$. | 9. $z^2 + 7z - 44 = 0$. |
| 10. $x^2 - 6x - 16 = 0$. | 11. $x^2 - 5x - 36 = 0$. |
| 12. $3y^2 + 36y = -108$. | 13. $5y^2 + 25y = 420$. |
| 14. $12z^2 - 5z - 2 = 0$. | 15. $4 - 4z = 3z^2$. |
| 16. $4x^2 + 13x + 3 = 0$. | 17. $4x^2 - 3 = -11x$. |

301.—SOLUTION BY COMPLETING THE SQUARE.

18. Find the values of x in $x^2 + 10x = 39$.

We add 25, the square of $\frac{1}{2}$ of 10, to the first member, to make it a perfect square (157); we also add it to the second member to preserve the equality; we thus obtain (2). Now extracting the square roots of both sides of (2) we obtain (3), from which the roots are readily found.

$$x^2 + 10x + 25 = 39 + 25.$$

$$x^2 + 10x + 25 = 64 \dots \dots (2)$$

$$x + 5 = \pm 8 \dots (3)$$

$$x = -5 \pm 8.$$

$$x = -5 + 8 = +3.$$

$$x = -5 - 8 = -13.$$

19. Find the values of x in $x^2 - 7x = -10$.

Complete the square by add-

ing $(\frac{7}{2})^2$, or $\frac{49}{4}$, to both sides,

Reducing the second side,

Extracting the square roots,

Whence,

$$x^2 - 7x + \frac{49}{4} = -10 + \frac{49}{4}$$

$$x^2 - 7x + \frac{49}{4} = \frac{9}{4}$$

$$x - \frac{7}{2} = \pm \frac{3}{2}$$

$$x = +\frac{7}{2} \pm \frac{3}{2}$$

$$= +5, \text{ and } +2.$$

20. Solve $5x^2 + 8x - 21 = 0$.

Transpose 21, and divide by 5,

$$x^2 + \frac{8}{5}x = \frac{21}{5}$$

Add $(\frac{1}{2} \text{ of } \frac{8}{5})^2$, or $\frac{16}{25}$,

$$x^2 + \frac{8}{5}x + \frac{16}{25} = \frac{21}{5} + \frac{16}{25}$$

Reduce the second side,

$$x^2 + \frac{8x}{5} + \frac{16}{25} = \frac{121}{25}$$

Extract the square root,

$$x + \frac{4}{5} = \pm \frac{11}{5}$$

Whence,

$$x = -\frac{4}{5} \pm \frac{11}{5}$$

$$= +\frac{7}{5}, \text{ and } -3.$$

Rule I.—Reduce the equation to the form of $x^2 + px = q$.

II.—Complete the square by adding the square of one-half the coefficient of x , $(\frac{p}{2})^2$, to both sides.

III.—Extract the square root of each side of the resulting equation.

IV.—Reduce the resulting simple equation.

Solve:

21. $x^2 + 6x = 16.$

22. $x^2 + 8x = 9.$

23. $x^2 - 8x = 20.$

24. $x^2 - 10x = 24.$

25. $x^2 + 12x = 13.$

26. $x^2 - 14x = 121.$

27. $x^2 + 3x = 4.$

28. $x^2 + 5x = 14.$

29. $x^2 - 7x = 30$. 30. $x^2 - 15x = 100$.
 31. $x^2 + 10x = -9$. 32. $x^2 + 18x = -17$.
 33. $x^2 - 22x = -40$. 34. $x^2 - 11x = -27$.
 35. $x^2 + 13x = -42$. 36. $x^2 - 19x = -78$.
 37. $3x^2 - 2x = 133$. 38. $2x^2 - 5x = 12$.
 39. $2x^2 + 3x = 65$. 40. $3x^2 + 2x = 85$.
 41. $6x^2 - 37x + 57 = 0$. 42. $7x^2 - 5x = 522$.
 43. $4x^2 + 3x - 115 = 0$. 44. $12x^2 - 187x + 481 = 0$.
 45. $\frac{2x^2}{3} - \frac{5x}{2} = \frac{2}{3}$. 46. $\frac{x}{4} - \frac{44}{x-2} = 4$.
 47. $20x^{-2} = 3 - 7x^{-1}$. 48. $2x - 7 + 3x^{-1} = 0$.
 49. $3x + 2\sqrt{x} = 16$. 50. $\sqrt{x} + \sqrt{10-x} = 3\sqrt{2}$.
 51. $x^2 + 2ax = b^2 + 2ab$. 52. $x^2 - 4cx = 9n^2 - 12cn$.
 53. $x^2 - \left(\frac{a^2 + b^2}{ab}\right)x = -1$. 54. $x^2 - \left(\frac{a^2 - b^2}{a}\right)x = b$.

302.—SOLUTION BY A FORMULA.

55. Solve $ax^2 + bx = c$ (1)

This equation may be solved by the method of Art. 301, but for the purpose of illustrating what is known as the "Hindoo Method," we give the following solution:

$$(1) \times 4a, \quad 4a^2x^2 + 4abx = 4ac.$$

Add b^2 to complete the square,

$$4a^2x^2 + 4abx + b^2 = b^2 + 4ac$$

Extract square root,

$$2ax + b = \pm \sqrt{b^2 + 4ac}.$$

Whence,

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}. \quad (2)$$

56. Solve $2x^2 - 7x = -3$.

Comparing this equation with (1), we have $a = 2$, $b = -7$, $c = -3$; substituting these values in (2),

$$x = \frac{7 \pm \sqrt{49 - 24}}{4} = \frac{7 \pm \sqrt{25}}{4} = \frac{7 \pm 5}{4} = 3 \text{ or } \frac{1}{2}.$$

57. Solve $\frac{x}{2} = \frac{7}{6} - \frac{2x^2}{3}$.

Clearing of fractions, and reducing to the form of (1),

$$4x^2 + 3x = 7.$$

Here, $a = 4$, $b = 3$, $c = 7$. Hence, substituting in (2),

$$x = \frac{-3 \pm \sqrt{9+112}}{8} = \frac{-3 \pm \sqrt{121}}{8} = \frac{-3 \pm 11}{8} = +1 \text{ or } -\frac{7}{4}.$$

Rule.—Reduce the equation to the form of (1). The value of x is then a fraction, whose numerator is the coefficient of x with its sign changed, plus or minus the square root of the sum of the square of said coefficient, and four times the product of the second member by the coefficient of x^2 ; and whose denominator is twice the coefficient of x^2 .

Solve:

58. $3x^2 + 5x = 42$.

59. $7x^2 + 4x = 51$.

60. $5x^2 + 8x = 21$.

61. $3x^2 + 25x = -8$.

62. $5x^2 - 46x = -9$.

63. $10x^2 - 23x = -12$.

64. $\frac{x^2}{2} + \frac{x}{3} = -\frac{1}{24}$.

65. $\frac{4x^2}{3} - \frac{17}{2} - \frac{x}{3} = 0$.

66. $6x + \frac{35-3x}{x} = 44$.

67. $4x - \frac{14-x}{x+1} = 14$.

68. $\frac{x}{5-x} - \frac{5-x}{x} = \frac{15}{4}$.

69. $\frac{7}{x^2-4} - \frac{3}{x+2} = \frac{22}{5}$.

70. $\frac{8x}{x+2} - 6 = \frac{20}{3x}$.

71. $\frac{3x-7}{x} + \frac{4x-10}{x+5} = 3\frac{1}{2}$.

72. $\frac{40}{x-5} - 13 = -\frac{27}{x}$.

73. $\frac{2x}{x-4} + \frac{2x-5}{x-3} = 8\frac{1}{3}$.

74. $\frac{x + \sqrt{x}}{x - \sqrt{x}} = \frac{2}{3}$.

75. $\frac{x - \sqrt{x+1}}{x + \sqrt{x+1}} = \frac{5}{11}$.

$$76. \frac{2x-3}{3x-5} + \frac{3x-5}{2x-3} = \frac{5}{2}.$$

$$77. \frac{3x-2}{2x-5} - \frac{2x-5}{3x-2} = \frac{8}{3}.$$

$$78. \frac{x+3}{x+2} - \frac{2x-3}{x-1} + \frac{x-3}{x-2} = 0.$$

$$79. 10(2x+3)(x-3) + (7x+3)^2 = 20(x+3)(x-1).$$

$$80. x^2 - 2ax + a^2 - b^2 = 0.$$

$$81. acx^2 - (bc - ad)x = bd.$$

$$82. \frac{a}{x-a} = \frac{2c}{x-c} - \frac{b}{x-b}.$$

$$83. \frac{x+a}{x-a} + \frac{x+b}{x-b} + \frac{x+c}{x-c} = 3.$$

$$84. x^2 - \frac{2(a-b)}{a+b} x = \frac{4ab}{(a+b)^2}.$$

$$85. x^2 - \frac{2p\sqrt{q}}{p-q} x = -\frac{pq}{p-q}.$$

$$86. a^2 + b^2 - 2bx + x^2 = \frac{m^2 x^2}{n^2}.$$

FORMATION OF EQUATIONS.

303. *Given the two roots of an equation to find the equation.*

EXERCISE LXVIII.

1. Find the equation whose roots are 5 and 8.

Since $x = 5$, $x - 5 = 0, \dots\dots\dots (1)$

and since $x = 8$, $x - 8 = 0, \dots\dots\dots (2)$

(1) $\times$ (2), $x^2 - 13x + 40 = 0$, Ans.

2. Find the equation whose roots are -10 and 6 .

Since $x = -10$, $x + 10 = 0$, ... (1)

and since $x = 6$, $x - 6 = 0$, ... (2)

(1) $\times$ (2), $x^2 + 4x - 60 = 0$, Ans.

3. Find the equation whose roots are r and r' .

Since $x = r$, $x - r = 0$, ... (1)

and since $x = r'$, $x - r' = 0$, ... (2)

(1) $\times$ (2), $x^2 - (r + r')x + rr' = 0$.

Rule.—*Subtract each root from x and place the product of the two remainders equal to 0; or*

For the first term of the equation take x^2 ; for the second term, take x with a coefficient equal to minus the sum of the two roots; and for the third term take the product of the two roots.

Find the equation whose roots are:

4. 5 and 9 . 5. $a + b$ and $+ a - b$.

6. 12 and $- 7$. 7. $a + b$ and $- a + b$.

8. $- 10$ and $+ 3$. 9. $3c + 1$ and $3c - 1$.

10. $- 5\frac{1}{2}$ and $- 6\frac{1}{2}$. 11. $3 + \sqrt{2}$ and $3 - \sqrt{2}$.

12. $+\frac{a}{b}$ and $-\frac{b}{a}$. 13. $\frac{a+b}{2}$ and $\frac{a-b}{2}$.

14. $+\frac{3}{4}a$ and $-\frac{2}{3}a$. 15. $\frac{a+b\sqrt{-1}}{2}$ and $\frac{a-b\sqrt{-1}}{2}$.

16. $\frac{a + \sqrt{4b + a^2}}{2}$ and $\frac{a - \sqrt{4b + a^2}}{2}$.

17. $-\frac{b + \sqrt{b^2 - 4ac}}{2a}$ and $-\frac{b - \sqrt{b^2 - 4ac}}{2a}$.

CHARACTER OF THE ROOTS.

304. The character of the roots of a quadratic equation may be determined by inspection.

305.—THE FOUR FORMS.

Every quadratic equation may be reduced to one of the four following forms:

$$x^2 + px - q = 0 \dots\dots\dots (1)$$

$$x^2 - px - q = 0 \dots\dots\dots (2)$$

$$x^2 + px + q = 0 \dots\dots\dots (3)$$

$$x^2 - px + q = 0 \dots\dots\dots (4)$$

In each of these,

1^o, p , with its sign changed, is the sum of the two roots; and

2^o, q , with its sign, is the product of the two roots.

306.—SIGNS OF THE ROOTS.

For brevity, denote the roots by r and r' .

In the first form, since $r \times r' = -q$, one of the roots, as r , is positive and the other (r') negative. Again, since $r - r' = -p$, the negative root is numerically the greater.

Thus, the roots of $x^2 + 5x - 14 = 0$, are $+2$ and -7 .

In the second form, since $r \times r' = -q$, one of the roots, as r , is positive and the other (r') is negative. Again, since $r - r' = +p$, the positive root is numerically the greater.

Thus, the roots of $x^2 - 3x - 40 = 0$, are $+8$ and -5 .

In the third form, since $r \times r' = +q$, the roots have like signs, and since $-r - r' = -p$, both roots are negative.

Thus, the roots of $x^2 + 5x + 6$, are -2 and -3 .

In the fourth form, since $r \times r' = +q$, the roots have like signs, and since $+r + r' = +p$, both roots are positive.

Thus, the roots of $x^2 - 11x + 28$, are $+4$ and $+7$.

307.—VALUES OF THE ROOTS.

$$\text{In the first form..} \begin{cases} r = -\frac{p}{2} + \sqrt{q + \frac{p^2}{4}}, \\ r' = -\frac{p}{2} - \sqrt{q + \frac{p^2}{4}}. \end{cases} \dots\dots (1)$$

$$\text{In the second form} \begin{cases} r = \frac{p}{2} + \sqrt{q + \frac{p^2}{4}}, \\ r' = \frac{p}{2} - \sqrt{q + \frac{p^2}{4}}. \end{cases} \dots\dots (2)$$

$$\text{In the third form} \begin{cases} r = -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}, \\ r' = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}. \end{cases} \dots\dots (3)$$

$$\text{In the fourth form} \begin{cases} r = \frac{p}{2} + \sqrt{\frac{p^2}{4} - q}, \\ r' = \frac{p}{2} - \sqrt{\frac{p^2}{4} - q}. \end{cases} \dots\dots (4)$$

308. UNEQUAL ROOTS. The roots of an equation of the first or of the second form are unequal; for, in either of these

$$r - r' = 2\sqrt{q + \frac{p^2}{4}}.$$

In an equation of the third or of the fourth form

$$r - r' = 2\sqrt{\frac{p^2}{4} - q};$$

that is, the roots are unequal if $\frac{p^2}{4}$ is greater or less than q .

309. EQUAL ROOTS. The roots of an equation of the third or of the fourth form are equal if $\frac{p^2}{4}$ is equal to q .

310. REAL ROOTS. The roots of an equation of the first or of the second form are real; for, in these forms, $\frac{p^2}{4} + q$ is positive.

The roots of an equation of the third or of the fourth form are real if $\frac{p^2}{4}$ is not less than q .

311. IMAGINARY ROOTS. The roots of an equation of the third or of the fourth form are imaginary if $\frac{p^2}{4}$ is less than q ; for in this case, $\frac{p^2}{4} - q$ is negative (250).

312. *Imaginary roots indicate incompatible conditions.*

$$\text{For, (324, K.), } \left(\frac{a+b}{2}\right)^2 > ab \dots (1)$$

That is, the square of one-half of the sum of two numbers is greater than their product.

Now, in (3) and (4) the roots are imaginary

$$\text{when } \left(\frac{p}{2}\right)^2 < q$$

$$\text{or when } \left(\frac{r+r'}{2}\right)^2 < rr'.$$

which, by (1), is impossible.

Hence, imaginary roots arise from ascribing to the product of two numbers a greater value than the square of one half of their sum.

Thus, if the sum of two numbers is 8, their greatest possible product is $\left(\frac{8}{2}\right)^2$; or 16. Hence, the conditions of the following problem are incompatible, viz.:

Find two numbers whose sum is 8 and whose product is 17.

As no real numbers will satisfy the conditions, we will obtain imaginary numbers for the answer by solving the problem. Thus,

Let x = one of the numbers,

then will $8 - x$ = the other.

Hence, $x(8-x) = 17$,

that is, $x^2 - 8x = -17$,

whence $x = 4 \pm \sqrt{-1}$.

HIGHER EQUATIONS.

313. There are many higher equations which may be solved by processes similar to those given in the preceding articles.

EXERCISE LXIX.

1. Solve $x^4 - 10x^2 = -9$.

Since x^4 is the square of x^2 , we may complete the square and proceed as in Art. 301. Thus:

Complete the square, $x^4 - 10x^2 + 25 = -9 + 25 = 16$.

Extract square root, $x^2 - 5 = \pm 4$.

Transpose, etc., $x^2 = 9$ or 1 .

Extract square root, $x = \pm 3$ or ± 1 , Ans.

2. Solve $x^{\frac{6}{5}} - 9x^{\frac{3}{5}} = -8$.

Since $x^{\frac{6}{5}}$ is the square of $x^{\frac{3}{5}}$, this equation may be solved in a similar manner.

Complete the square, $x^{\frac{6}{5}} - 9x^{\frac{3}{5}} + (\frac{9}{2})^2 = -8 + \frac{81}{4} = \frac{49}{4}$.

Extract square root, $x^{\frac{3}{5}} - \frac{9}{2} = \pm \frac{7}{2}$.

Transpose, etc., $x^{\frac{3}{5}} = 8$ or 1 .

Extract cube root, $x^{\frac{1}{5}} = 2$ or 1 .

Raise to 5th power, $x = 32$ or 1 , Ans.

NOTE.—This equation has six roots; only the two *real* roots are here given.

3. Solve $x + 3\sqrt{x} = 40$.

Writing the radical with a fractional exponent,

$$x + 3x^{\frac{1}{2}} = 40.$$

Complete the square, $x + 3x^{\frac{1}{2}} + \left(\frac{3}{2}\right)^2 = \frac{169}{4}$.

Extract square root, $\sqrt{x} + \frac{3}{2} = \pm \frac{13}{2}$.

Transpose, etc., $\sqrt{x} = 5$ or -8 .

Squaring, $x = 25$ or 64 .

4. Solve $(x^2 - 5x)^2 - 8(x^2 - 5x) = 84$.

Since $(x^2 - 5x)^2$ is the square of $x^2 - 5x$, we may proceed as before.

Complete the square, $(x^2 - 5x)^2 - 8(x^2 - 5x) + 16 = 100$.

Extract square root, $x^2 - 5x - 4 = \pm 10$.

Transpose, etc., $x^2 - 5x = 14$, or -6 .

That is, $x^2 - 5x = 14$, and $x^2 - 5x = -6$.

Whence, $x = 7$ or -2 , and $x = 2$ or 3 .

5. Solve $2x^2 + 3\sqrt{2x^2 + 7} = 33$.

NOTE.—In solving equations of this form, add such a number to both sides, that the expression without the radical in the first member may be the same as that within, or some multiple of it.

Add 7 to both sides, $(2x^2 + 7) + 3\sqrt{2x^2 + 7} = 40$.

Complete the square, $(2x^2 + 7) + 3\sqrt{2x^2 + 7} + (\frac{3}{2})^2 = \frac{169}{4}$.

Extract the square root, $\sqrt{2x^2 + 7} + \frac{3}{2} = \pm \frac{13}{2}$.

Transpose, etc., $\sqrt{2x^2 + 7} = 5$, or -8 .

Squaring, $2x^2 + 7 = 25$, and $2x^2 + 7 = 64$.

Whence, $x = \pm 3$; and $x = \pm \sqrt{\frac{57}{2}}$.

314. Only certain forms of equations of the fourth degree can be solved as quadratics; of these are those of the following form:

$$x^4 + 2ax^3 + (a^2 + 2b)x^2 + 2abx = c.$$

If to the first side of this and similar equations we add b^2 , or the square of the quotient obtained by dividing the coefficient of x by the coefficient of x^3 , it will be a perfect square.

6. Solve $x^4 + 10x^3 + 17x^2 - 40x = 84$.

Divide the coefficient of x , -40 , by that of x^3 , $+10$, and we obtain -4 . Add the square of -4 to both members, and we have

$$x^4 + 10x^3 + 17x^2 - 40x + 16 = 84 + 16 = 100.$$

Extract the square root of both sides,

$$x^2 + 5x - 4 = \pm 10,$$

Transpose, etc., $x^2 + 5x = 14$, and $x^2 + 5x = -6$

Whence, $x = -7$ or $+2$, and $x = -2$ or -3 , Ans.

Solve:

7. $x^4 - 14x^2 = -40$.

8. $2\sqrt{x} + \frac{2}{\sqrt{x}} = 5$.

9. $x^{10} + 31x^5 = 32$.

10. $\sqrt[3]{x} + \frac{5}{2\sqrt[3]{x}} = 3\frac{1}{2}$.

11. $x^3 + 14x^{\frac{3}{2}} = 1107$.

12. $x^2 - 2x + 3 = 2\sqrt{x^2 - 2x + 2}$.

13. $5x^{\frac{1}{2}} + x^{\frac{1}{4}} = 22$.

14. $x^2 + 5x + 4 = 5\sqrt{x^2 + 5x + 28}$.

15. $3x^{\frac{3}{2}} - 4x^{\frac{3}{4}} = 7$.

16. $3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2$.

17. $x^{-1} + x^{-\frac{1}{2}} = 6$.

18. $(x + 5)(x - 2) = -3\sqrt{x^2 + 3x}$.

19. $3x + 2\sqrt{x} = 1$.

20. $(x^2 + 2x)^2 - 3(x^2 + 2x) + 2 = 0$.

21. $(x^2 - 5x)^2 - 8(x^2 - 5x) = 84$.

22. $(3x^2 - 10x + 5)^2 - 8(3x^2 - 10x + 5) = -12$.

Solve:

$$23. x^4 - 12x^3 + 34x^2 + 12x = 35.$$

$$24. x^4 - 10x^3 + 35x^2 - 50x = -24.$$

$$25. x^4 + 6x^3 + 5x^2 - 12x = 12.$$

$$26. x^4 + 3x^3 + 2x^2 - \frac{3}{8}x = \frac{5}{4}.$$

PROBLEMS.

315.—EXERCISE LXX.

1. Find a number such that, if 6 times the number be added to its square, the sum shall be 55.

Let x = the number,
 then $x^2 + 6x = 55$,
 whence $x = -3 \pm 8, = 5$ or -11 .

2. Divide the number 24 into two such parts, that their product shall be 95.

Let x = one of the parts,
 then $24 - x$ = the other.
 and $x(24 - x) = 95$.
 whence, $x = 19$ or 5 ,
 and $24 - x = 5$ or 19 .

Divide the number 20 into two parts:

3. Such that their product shall be 51.

4. Such that their product shall be 91.

5. Such that their product shall be 105.

6. Find a number such that, if 15 be added to its square, the sum shall be 8 times the number.

Let x = the number,
 then $x^2 + 15 = 8x$.
 whence, $x = 5$ or 3 .

Find a number such that, if 36 be added to its square:

7. The sum shall be 15 times the number.

8. The sum shall be 13 times the number.

9. The sum shall be 11 times the number.

10. Find two numbers whose difference is 7, and whose sum multiplied by the less is 85.

Let $x =$ the less,
 then $x + 7 =$ the greater,
 and $2x + 7 =$ their sum.

Hence, $x(2x + 7) = 85$.

Whence, $x = 5$ or $-8\frac{1}{2}$, the less,
 and $x + 7 = 12$ or $-1\frac{1}{2}$, the greater.

11. A person bought a number of oxen for £80, and if he had bought 4 more for the same money, the price of each would have been £1 less: how many did he purchase?

Let $x =$ the number of oxen.

then $\frac{80}{x} =$ the price paid for each.

and $\frac{80}{x+4} = \begin{cases} \text{the price that would have been paid for each,} \\ \text{had 4 more been bought.} \end{cases}$

Hence, $\frac{80}{x} = \frac{80}{x+4} + 1$.

Whence, $x = 16$ or -20 .

The number bought was therefore 16.

12. A person bought a number of sheep for \$120; and if the price per head had been \$1 less, the number bought would have been 10 more. Required the price paid for each sheep.

13. A man traveled 132 miles; if his rate per hour had been 2 miles more he would have gone the distance in 11 hours less time. Required his rate per hour.

14. If to the product of two consecutive numbers their sum be added, the result will be 131. What are the numbers?

15. The sum of the squares of two consecutive numbers is 211 more than their product. What are the numbers?

16. Of four consecutive numbers, if the first be multiplied by the fourth and the second by the third, the sum of the products will be 80 more than the sum of the numbers. What are the numbers?

17. The length of a rectangular field exceeds its breadth by 2 rods, and the area is $1\frac{2}{5}$ acres; find the length of the sides.

18. The length of a rectangular field is twice its breadth; if each of its dimensions were increased by 3 chains, it would then contain 17 acres. Find the length of its sides.

19. A plot of ground is 18 feet by 12, around which is a walk of uniform width, whose area is equal to that of the plot. Find the width of the walk.

20. The area of a square is doubled by increasing its length by 15 inches and decreasing its breadth by 2 inches. Find its side.

21. Divide 30 into two parts so that the sum of the two fractions formed by dividing each part by the other may be $2\frac{1}{6}$.

22. What number is that which, being added, first to the numerator and then to the denominator of $\frac{2}{5}$, the former result is 4 times the latter.

23. A fraction is equivalent to $\frac{3}{4}$, and is increased by

$\frac{1}{24}$ when 4 is added to each of its terms. Find the fraction.

24. A can do a work in 2 hours' less time than B can do it, and together they can do it in $3\frac{3}{7}$ hours. How long will it take each alone to do it?

25. A rope weighs 84 pounds; if it were two feet longer each foot would weigh $\frac{1}{10}$ of a pound less. Find the length of the rope.

26. There are 24 yards in two pieces of cloth. The whole cost 18s., and each yard of either piece cost as many pence as there are yards in the other piece; how many yards in each piece?

27. There are two cubical blocks of stone, one of which contains 117 cubic feet more than the other, and the edge of the larger is 3 ft. more than that of the smaller. Required the dimensions of each.

28. A and B hired a pasture into which A put 4 horses, and B as many as cost him 18 shillings a week. Afterwards B put in 2 more horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired?

29. Two persons, A and B, jointly invested \$500 in business; A let his money remain 5 months, and B only 2, and each received back \$450, capital and profit. How much did each advance?

30. A traveled from C towards D at the rate of 7 miles a day. After he had gone 32 miles, B set out from D towards C, and went every day $\frac{1}{15}$ of the whole journey; and after he had traveled as many days as he went miles in one day, he met A. Find the distance from C to D,

31. If it requires 900 rails to fence an acre, what is the area of a square field which contains as many acres as there are rails in the fence that encloses it?

EQUATIONS CONTAINING TWO OR MORE UNKNOWN NUMBERS.

EXERCISE LXXI.

BY SUBSTITUTION.

316.—1. Find the value of $\begin{cases} x + y = 5 \dots (1) \\ x^2 + y^2 = 13 \dots (2) \end{cases}$
 x and y in

By (1),

$$x = 5 - y$$

Substitute in (2),

$$(5 - y)^2 + y^2 = 13$$

Whence,

$$y = 3 \text{ or } 2.$$

Substitute in (1),

$$x = 2 \text{ or } 3.$$

2. Find the value of $\begin{cases} 3x^2 - 5y^2 = 55 \dots (1) \\ xy = 10 \dots (2) \end{cases}$
 x and y in

By (2),

$$x = \frac{10}{y}$$

Substitute in (1)

$$\frac{300}{y^2} - 5y^2 = 55$$

Reduce, divide by 5,

$$y^4 + 11y^2 = 60$$

Whence,

$$y = \pm 2 \text{ or } \pm \sqrt{-15}.$$

Substitute in (2),

$$x = \pm 5 \text{ or } \pm \frac{2}{3}\sqrt{-15}.$$

Find the values of x and y in:

$$\begin{aligned} 3. \quad 4x^2 + 7y^2 &= 148, \\ 3x^2 - y^2 &= 11. \end{aligned}$$

$$\begin{aligned} 4. \quad x + y &= 7, \\ x^2 + 2y^2 &= 34. \end{aligned}$$

$$\begin{aligned} 5. \quad x^2 + y^2 &= 65, \\ xy &= 28. \end{aligned}$$

$$\begin{aligned} 6. \quad x^2 + xy &= 15, \\ xy - y^2 &= 2. \end{aligned}$$

$$\begin{aligned} 7. \quad x^2 - 4y^2 &= 9, \\ xy + 2y^2 &= 3. \end{aligned}$$

$$\begin{aligned} 8. \quad 7x^2 - 8xy &= 159, \\ 5x + 2y &= 7. \end{aligned}$$

NOTE I.—The examples under the two succeeding articles may also be solved by the method of substitution. The student should at least solve a sufficient number of them in this manner to make him familiar with the process.

II.—Special methods, by reduction of the equations, often afford more elegant solutions than the general method by substitution.

BY REDUCTION.

317. Reduction of equations consists in transforming them into simpler equations.

318. I.—*When the side containing the unknown number can be made a perfect square.*

9. Find the values $x^2 + y^2 = 41 \dots\dots (1)$

of x and y in $xy = 20 \dots\dots (2)$

$(1) + (2) \times 2, \quad x^2 + 2xy + y^2 = 81 \dots\dots (3)$

$(1) - (2) \times 2, \quad x^2 - 2xy + y^2 = 1 \dots\dots (4)$

$\sqrt{(3)}, \quad x + y = \pm 9 \dots\dots (5)$

$\sqrt{(4)}, \quad x - y = \pm 1 \dots\dots (6)$

We thus obtain two simple equations, (5) and (6), which express the same conditions, though under different forms, that (1) and (2) do.

Combining (5) and (6), we get $x = \pm 5, y = \pm 4$.

10. Find the values $\frac{1}{x} + \frac{1}{y} = \frac{3}{4} \dots\dots (1)$

of x and y in $\frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{16} \dots\dots (2)$

$(2) \times 2 - (1)^2, \quad \frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{16} \dots\dots (3)$

$\sqrt{(3)}, \quad \frac{1}{x} - \frac{1}{y} = \pm \frac{1}{4} \dots\dots (4)$

By combining (1) and (4), $x = 2$ or $4, y = 4$ or 2 .

11. Find the values
of x and y in

$$x + y = 13 \dots\dots (1)$$

$$xy = 42 \dots\dots (2)$$

$$(1)^2 - (2) \times 4,$$

$$\sqrt{(3)},$$

Combine (1) and (4),

$$x^2 - 2xy + y^2 = 1 \dots\dots (3)$$

$$x - y = \pm 1 \dots\dots (4)$$

$$x = 7 \text{ or } 6, y = 6 \text{ or } 7.$$

Find the values of x and y in

$$12. \begin{aligned} x + y &= 7, \\ xy &= 10. \end{aligned}$$

$$13. \begin{aligned} x - y &= 6, \\ xy &= 16. \end{aligned}$$

$$14. \begin{aligned} x + y &= 18, \\ xy &= 77. \end{aligned}$$

$$15. \begin{aligned} x - y &= 5, \\ xy &= 84. \end{aligned}$$

$$16. \begin{aligned} x^2 + y^2 &= 100, \\ x + y &= 14. \end{aligned}$$

$$17. \begin{aligned} x^2 + y^2 &= 325, \\ x - y &= 5. \end{aligned}$$

$$18. \begin{aligned} x^2 + y^2 &= 676, \\ x + y &= 34. \end{aligned}$$

$$19. \begin{aligned} x^2 + y^2 &= 169, \\ x - y &= 7. \end{aligned}$$

$$20. \begin{aligned} \frac{1}{x^2} + \frac{1}{y^2} &= \frac{13}{36}, \\ \frac{1}{xy} &= \frac{1}{6}. \end{aligned}$$

$$21. \begin{aligned} \frac{1}{x^2} + \frac{1}{y^2} &= \frac{29}{100}, \\ \frac{1}{xy} &= \frac{1}{10}. \end{aligned}$$

$$22. \begin{aligned} \frac{1}{x^2} + \frac{1}{y^2} &= 41, \\ \frac{1}{x} + \frac{1}{y} &= 9. \end{aligned}$$

$$23. \begin{aligned} \frac{1}{x^2} + \frac{1}{y^2} &= 85, \\ \frac{1}{x} - \frac{1}{y} &= 1. \end{aligned}$$

$$24. \begin{aligned} x^4 + y^4 &= 17, \\ xy &= 2. \end{aligned}$$

$$25. \begin{aligned} x^4 + y^4 &= 97, \\ x^2 + y^2 &= 13. \end{aligned}$$

$$26. \begin{aligned} x^6 + y^6 &= 15689, \\ xy &= 10. \end{aligned}$$

$$27. \begin{aligned} x^6 + y^4 &= 985, \\ x^3 y^2 &= 432. \end{aligned}$$

320. III.—*When the equations are homogeneous and of the second degree.*

42. Find the values $2y^2 - 4xy + 3x^2 = 17 \dots (1)$
of x and y in $y^2 - x^2 = 16 \dots (2)$

In (1) and (2) make $y = vx$, $x^2(2v^2 - 4v + 3) = 17 \dots (3)$

$x^2(v^2 - 1) = 16 \dots (4)$

From (3), $x^2 = \frac{17}{2v^2 - 4v + 3}$; from (4), $x^2 = \frac{16}{v^2 - 1} \dots (5)$

$\therefore \frac{17}{2v^2 - 4v + 3} = \frac{16}{v^2 - 1}$. Whence, $v = \frac{5}{3}$ or $\frac{13}{5}$.

In (5) make $v = \frac{5}{3}$; whence, $x = \pm 3, y = \pm 5$.

In (5) make $v = \frac{13}{5}$; whence, $x = \pm \frac{5}{3}, y = \pm \frac{13}{3}$.

Find the values of x and y in:

43. $x^2 + 3xy = 54, xy + 4y^2 = 115$.

44. $x^2 + xy + 4y^2 = 6, 3x^2 + 8y^2 = 14$.

45. $6x^2 - 5xy + 2y^2 = 12, 3x^2 + 2xy - 3y^2 = -3$.

46. $2x^2 - 7xy - 2y^2 = 5, 3xy - x^2 + 6y^2 = 44$.

PROBLEMS.

EXERCISE LXXII.

321. The difference of two numbers is 5, and the sum of their squares is 193. Find the numbers.

Let x and y be the numbers.

Then, $x - y = 5 \dots (1)$

and $x^2 + y^2 = 193 \dots (2)$

From (1), $x = 5 + y$

Substitute in (2), $(5 + y)^2 + y^2 = 193$

Whence, $y = +7$ or -12 .

Substitute in (1), $x = +12$ or -7 .

2. Divide 40 into two such parts that the sum of their squares shall be 818.

3. What two numbers are those whose difference multiplied by the less produces 42, and by their sum, 133?

4. What two numbers are those whose sum multiplied by the greater gives 144, and whose difference multiplied by the less gives 14?

5. The difference of the cubes of two numbers is 19, and their product multiplied by their difference is 6; find the numbers.

6. What number is that which is equal to twice the product of its two digits, and if 27 be added to it, the order of the digits will be reversed?

Let x and y be the digits
 Then $10x + y = \text{the number.}$
 Hence, $10x + y = 2xy \dots\dots\dots (1)$
 and $10x + y + 27 = 10y + x \dots\dots (2)$
 Reduce (2), $y = x + 3 \dots\dots\dots (3)$
 Substitute this value of y in (1),
 $10x + (x + 3) = 2x(x + 3)$
 Or, $11x + 3 = 2x^2 + 6x$
 Whence, $x = 3$, or $-\frac{1}{2}$.
 Substitute in (3), $y = 6$, or $+2\frac{1}{2}$.
 Therefore the number is 36.

7. What number is that which, when multiplied by its units' digit, the product is 69; and when multiplied by its tens' digit, the product is 46?

8. There are four consecutive numbers, of which if the first two be taken for the digits of a number, that number is the product of the other two. Find the four numbers.

9. The fore wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yards; but if the circumference of each wheel be increased one yard, it will make only 4 revolutions more than the hind wheel, in the same distance; required the circumference of each wheel.

Let x = circum. of hind wheel in yards,
and y = circum. of fore wheel in yards.

Then, $\frac{120}{x}$ = number of revolutions of hind wheel.

and $\frac{120}{y}$ = number of revolutions of fore wheel.

Then, by the $\left\{ \begin{array}{l} \frac{120}{y} = \frac{120}{x} + 6 \dots\dots\dots (1) \end{array} \right.$

conditions: $\left\{ \begin{array}{l} \frac{120}{y+1} = \frac{120}{x+1} + 4 \dots\dots\dots (2) \end{array} \right.$

$$(1) \times xy, \quad 120x = 120y + 6xy \dots\dots\dots (3)$$

$$(2) \times (y+1)(x+1), \quad 120x+120=120y+120+4xy+4x+4y+4 \dots\dots\dots (4)$$

$$\text{Or,} \quad xy = 29x - 31y - 1 \dots\dots\dots (5)$$

$$\text{From (3),} \quad xy = 20x - 20y \dots\dots\dots (6)$$

$$(5) - (6), \quad 0 = 9x - 11y - 1 \dots\dots\dots (7)$$

$$\text{Whence,} \quad x = \frac{11y+1}{9} \dots\dots\dots (8)$$

$$\text{Substitute in (6),} \quad \frac{11y^2+y}{9} = \frac{220y+20}{9} - 20y$$

$$\text{Whence,} \quad y = 4, \text{ or } -\frac{5}{11}.$$

$$\text{Substitute in (8),} \quad x = 5, \text{ or } -\frac{4}{9}.$$

Therefore the two circumferences are 4 and 5 yards.

10. Two men, A and B, traveled from C to D, a distance of 60 miles. They started at the same time, and A reached D 5 hours in advance of B; but if B's rate per hour had been 2 miles more and A's rate 1

mile less, then B would have reached D 2 hours in advance of A. Required the rates at which A and B traveled per hour.

11. It requires 200 rods of fence to enclose two square lots whose contents are 1300 square rods. Required one side of each lot.

12. A person bought a number of pigs for \$35; after losing two of them he sold the rest at $\$1\frac{1}{2}$ a head more than he gave for them, and by so doing gained \$1 by the transaction. Find the number of pigs he bought.

13. A gentleman sends a lad into the market to buy 12 cents worth of oranges. The lad having eaten a couple, the gentleman pays at the rate of a cent for fifteen more than the market price. How many did the gentleman receive?

14. There are three numbers, the difference of their differences is 8; their sum is 41; and the sum of their squares 699. Find the numbers.

Let x, y and z be the numbers.

Then $(x - y) - (y - z) = 8$

or, $x - 2y + z = 8 \dots\dots\dots (1)$

$x + y + z = 41 \dots\dots\dots (2)$

$x^2 + y^2 + z^2 = 699 \dots\dots\dots (3)$

(2) - (1), $3y = 33, \therefore y = 11.$

In (2) substitute 11 for y , $x + z = 30 \dots\dots (4)$

In (3) substitute 11 for y , $x^2 + z^2 = 578 \dots\dots (5)$

Combine (4) and (5), and we find

$$x = 23, z = 7.$$

15. There are three numbers, the difference of their differences is 1; the difference of the differences of their squares is 25; and the square of the intermediate

number is equal to the product of the other two. Find the numbers.

16. The sum of two numbers multiplied by their difference is 21, and the sum of their squares multiplied by their product is 290. Find the numbers.

Let x and y be the numbers.

$$\text{Then, } (x + y)(x - y) = x^2 - y^2 = 21 \dots\dots (1)$$

$$\text{and } (x^2 + y^2)xy = x^3y + xy^3 = 290 \dots\dots (2)$$

$$(1)^2 \times x^2y^2, \quad x^6y^2 - 2x^4y^4 + x^2y^6 = 441x^2y^2 \dots\dots (3)$$

$$(2)^2, \quad x^6y^2 + 2x^4y^4 + x^2y^6 = 84100 \dots\dots (4)$$

$$(4) - (3), \quad 4x^4y^4 = 84100 - 441x^2y^2$$

$$\text{Whence, taking the positive value, } xy = 10 \dots\dots (5)$$

$$\text{Substituting in (2), } x^2 + y^2 = 29 \dots\dots (6)$$

$$\text{Combining (5) and (6), } x = 5, y = 2.$$

17. A person bought two cubical stacks of hay for \$41, each of which cost as many nickels per solid yard as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each?

18. A grocer sold 80 pounds of tea and 100 pounds of coffee for \$65; and he sold 60 pounds more of coffee for \$20 than he did of tea for \$10. What was the price of each?

19. A and B together can do a certain work in 12 days; if A could do the work in 8 days less time, and B in 6 days less time, both together could do it in 8 days. How long will it take each alone to do the work?

20. The height of the walls of a room is 11 feet, and the length of the ceiling is 2 feet more than its width. The cost of plastering the walls and ceiling at \$ $\frac{1}{3}$ per

square yard, is \$38 $\frac{1}{2}$, no allowance being made for doors and windows. Find the length and width of the room.

21. A sets off from C to D, and B at the same time from D to C, and they travel uniformly; A reaches D 16 hours, and B reaches C 36 hours, after they have met on the road. Find in what time each has performed the journey.

22. A certain number of workmen can move a heap of stones in 8 hours from one place to another. If there had been 8 more workmen, and each workman had carried 5 lbs. less at a time, the whole work would have been completed in 7 hours. If, however, there had been 8 less workmen, and each had carried 11 lbs. more at a time, the work would have occupied 9 hours. Find the number of workmen and the weight which each carried at a time.

NOTE.—The longer side of a right-triangle is called the *hypotenuse* (H), and the other two sides, the *base* (B) and *perpendicular* (P).

$$\text{Formula: } H^2 = B^2 + P^2.$$

23. A cord 100 feet long is stretched over the limb of a tree in such a manner that while one of its ends reaches the ground directly under the limb, the other touches the ground at a distance of 40 feet from that point. Find the height of the limb.

24. A ladder 50 feet long is leaning against a wall so as to reach a window. Now if the bottom of the ladder be pulled out 16 feet farther from the wall, it will throw the top of it 8 feet below the window. Required the height of the window from the ground.

25. A and B are two towns on the bank of a stream, and are 42 miles apart. A man rows from A to B and back again, and finds, first, that he is 3 hrs. 12 min. longer upon the water than he would have been had there been no current; second, that he was 8 hours longer going from A to B than B to A. Find his rate of rowing, and the rate of the current.

26. Find the price of eggs per dozen when two more for 12 cents lowers the price one cent per dozen.

27. Two farmers drove to market 100 sheep between them, and returned with equal sums. If each of them had sold his sheep at the same price that the other did, the one would have returned with \$180 and the other with \$80. At what price per sheep did they sell, and how many sheep had each?

28. A ladder just reaches a window on one side of a street, and when turned about its foot, just reaches a window on the other side. If the two positions of the ladder are at right angles to each other, and the heights of the windows are 45 and 28 feet respectively, find the width of the street and the length of the ladder.

29. Find two numbers whose product equals the difference of their squares, and the sum of whose squares equals the difference of their cubes.

CHAPTER IX.

INEQUALITIES, ZERO AND INFINITY.

INEQUALITIES.

322. An **Inequality** is a statement in algebraic language that one quantity is greater or less than another.

Thus, $a > b$ is an inequality and indicates that a is greater than b , or that $a - b$ is positive.

Again, $a < b$ is an inequality and expresses that a is less than b , or that $a - b$ is negative.

The form $a > b > c$ indicates that b is less than a but greater than c , or that b is intermediate in value between a and c .

323. The fourth particular in which Algebra is an extension of Arithmetic (205) is the use of inequalities in deducing certain relations and properties of quantities.

324. The principles (207) applied to the solutions of equations may be applied to inequalities, except that if each side of an inequality have its sign changed, the sign $>$ will be reversed.

Thus,	$3 > 2 \dots\dots\dots (1)$
Multiply (1) by -1 ,	$-3 < -2 \dots\dots\dots (2)$
Square (2)	$9 > 4.$

EXERCISE LXXIII.

1. If $5x - 17 > 3x + 7$, show that $x > 12$.

Transpose,	$5x - 3x > 17 + 7.$
Reduce,	$2x > 24.$
Divide by 2,	$x > 12.$

Show that:

2. If $3x - 5 > 13$, then $x > 6$.

3. If $\frac{7x}{5} - \frac{5x}{3} > \frac{x}{3} - 3$, then $x < 5$.

4. If $2x - 5 > 25$, and $3x - 7 < 2x + 13$, then
 $x > 15$ and $x < 20$.

5. If $3x + 1 > 13 - x$, and $4x - 7 < 2x + 3$, then
 $5 > x > 3$.

6. If $2x + 4y > 30$, and $3x + 2y = 31$, then $x < 8$,
 $y > 3\frac{1}{2}$.

7. If $5x + 3y > 121$, and $7x + 4y = 168$, then $x < 20$,
 $y > 7$.

NOTE.—In the following exercises the letters a , b , c , etc., represent any positive, real and unequal numbers, unless otherwise stated.

8. Show that $a^2 + b^2 > 2ab$(A)

Evidently,

$(a - b)^2$ is positive.

Or,

$(a - b)^2 > 0$.

Expand,

$a^2 - 2ab + b^2 > 0$.

Transpose,

$a^2 + b^2 > 2ab$.

9. Show that the sum of any fraction and its reciprocal is greater than 2.

Divide both sides of (A) by ab .

10. Show that $a^3 + b^3 > a^2b + ab^2$(B)

(A) $\times a$,

$a^3 + ab^2 > 2a^2b$(1)

(A) $\times b$,

$a^2b + b^3 > 2ab^2$(2)

[(1) + (2)]

$a^3 + b^3 > a^2b + ab^2$.

11. Show that $a^2 + b^2 + c^2 > ab + ac + bc$(C)

By (A),

$a^2 + b^2 > 2ab$(1)

$a^2 + c^2 > 2ac$(2)

$b^2 + c^2 > 2bc$(3)

[(1) + (2) + (3)] $\div 2$, $a^2 + b^2 + c^2 > ab + ac + bc$.

12. Show that $a^3 + b^3 + c^3 > 3abc$(D)

Multiply (C) by $a + b + c$, and reduce.

13. Show that $\frac{r + r'}{2} > \sqrt{rr'}$(E)

In (A) make $r = a^2$, $r' = b^2$, and reduce.

14. Show that $\frac{r + r' + r''}{3} > \sqrt[3]{rr'r''}$(F)

In (D) make $r = a^3$, $r' = b^3$, $r'' = c^3$, and reduce.

15. Show that $\frac{a^2 + n}{2a} > \sqrt{n}$(G)

In (E) make $r = a$, $r' = \frac{n}{a}$, and reduce.

16. Show that $\frac{2a^3 + n}{3a^2} > \sqrt[3]{n}$(H)

In (F) make $r = r' = a$, $r'' = \frac{n}{a^2}$ and reduce.

17. Show that $\left(\frac{a + b}{2}\right)^2 > ab$(K)

18. Show that $\frac{r + r' + r'' + r'''}{4} > \sqrt[4]{rr'r''r'''}$. (L)

In (K) make $a = \frac{r + r'}{2}$ and $b = \frac{r'' + r'''}{2}$, and reduce.

19. If $a + b + c = m$, $ab + ac + bc = n$, and $abc = r$, show that

$$\begin{aligned} (1), \dots m^2 &> 3n; & (3), \dots n^3 &> 27r^2; \\ (2), \dots m^3 &> 27r; & (4), \dots (m^2 - 2n)^3 &> 27r^2. \end{aligned}$$

325. The principles of inequalities afford elegant methods of deducing many useful formulas. We pur-

pose now to show how they may be employed in obtaining an approximate square or cube root of a number.

326. An Approximate Root is a number which differs but little from the true root.

Thus, an approximate square root of 7 is 2, or (more approximate) 2.6, or (still more approximate) 2.64.

327. To find the *approximaté* square root of a number.

If $a > \sqrt{n}$, then $\frac{a^2 + n}{2a}$ is a more approximate value of $\sqrt{n}$ than a is.

For, since $a > \sqrt{n}$, $a^2 > n$, and $\frac{n}{a} < a \dots \dots \dots (1)$

Add a to both sides, reduce, $\frac{a^2 + n}{2a} < a \dots \dots \dots (2)$

But, (G), $\frac{a^2 + n}{2a} > \sqrt{n} \dots \dots \dots (3)$

Hence, $a > \frac{a^2 + n}{2a} > \sqrt{n}$.

That is, $\frac{a^2 + n}{2a}$ is intermediate in value between a and $\sqrt{n}$.

328. Hence, to find a more approximate value of $\sqrt{n}$, *substitute any approximate value of it for a in $\frac{a^2 + n}{2a}$, and reduce the result to its simplest form.*

EXERCISE LXXIV.

1. Find the approximate square root of 24.

Take $a = 5$ for 1^o approx. root.

Now substitute 5 for a in $\frac{a^2 + n}{2a}$ and we get the 2^o approx.

root viz.: $\frac{49}{10} = 4.9$.

Again, substitute 4.9 for a in $\frac{a^2 + 24}{2a}$ and we get the 3^o approx. root, viz.: $\frac{4801}{980} = 4.89896 +$.

The correct answer to eight decimal places is 4.89897949. Hence, the 3^o approx. root is true to four decimals.

Now the 4^o approx. root, which may be found by substituting the 3^o approx. root for a in $\frac{a^2 + 24}{2a}$, is still more nearly equal to the true answer. Continuing thus, the approximation may be carried to any degree of accuracy.

Find the 3^o approximate square root:

2. Of 2, taking $1\frac{1}{2}$ for the 1^o approx. root.
3. Of 26, taking 5 for the 1^o approx. root.
4. Of 10, taking 3 for the 1^o approx. root.
5. Find the approx. value of $\sqrt{1+x}$.

Take 1 for the 1^o approx. value.

Then, $\frac{1^2 + 1 + x}{2} = \frac{2+x}{2} = 2^{\text{o}} \text{ approx. value,}$

and $\left[\left(\frac{2+x}{2} \right)^2 + (1+x) \right] \div 2(1+x) = \frac{8+8x+x^2}{8+4x} =$

$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$ etc. = the 3^o approx. value,

which is true to four terms.

329. To find the approximate cube root of a number.

If $a > \sqrt[3]{n}$, then $\frac{2a^3 + n}{3a^2}$ is a more approximate value of $\sqrt[3]{n}$ than a is.

For, since $a > \sqrt[3]{n}$, $a^3 > n$ and $\frac{n}{a^2} < a$.

Add $2a$ to both sides, reduce, $\frac{2a^3 + n}{3a^2} < a$.

But (H), $\frac{2a^3 + n}{3a^2} > \sqrt[3]{n}$.

Hence, $a > \frac{2a^3 + n}{3a^2} > \sqrt[3]{n}$. See Art. 328.

That is, $\frac{2a^3 + n}{3a^2}$ is intermediate in value between a and $\sqrt[3]{n}$.

EXERCISE LXXV.

1. Find the approximate cube root of 9.

Take $a = 2$ for 1^o approx. root.

Now substitute 2 for a in $\frac{2a^3 + 9}{3a^2}$ and we

get the 2^o approx. root, viz.: $\frac{25}{12} = 2.083 +$.

Again, substitute $\frac{25}{12}$ for a in $\frac{2a^3 + 9}{3a^2}$ and we

get the 3^o approx. root, viz.: $\frac{23401}{11250} = 2.08008 +$, which

is the correct answer to five decimals.

Find the 3^o approx. cube root:

2. Of 3, taking $1\frac{1}{2}$ for the 1^o approx. root.

3. Of 2, taking $1\frac{1}{4}$ for the 1^o approx. root.

4. Of 11, taking $2\frac{1}{3}$ for the 1^o approx. root.

ZERO AND INFINITY.

330. The fifth particular in which Algebra is an extension of Arithmetic (**323**) is the extended signification and use of zero, or 0.

331. In Arithmetic, all zeros are regarded as being equal to each other, since each denotes none or nothing in comparison with a given finite quantity. But it does not follow from this that all zeros are equal in comparison with each other.

332. Finite Quantities are the ordinary quantities denoted by figures and letters.

333. Zero, denoted by 0, is the result of subtracting a finite quantity from itself.

$$\text{As, } a - a = 0, \quad nx - nx = 0.$$

334. Infinity, denoted by ∞ , is the reciprocal of 0.

$$\text{Thus, } \frac{1}{a - a} = \frac{1}{0} = \infty.$$

PRINCIPLES AND PROPERTIES.

(See Art. 345.)

335. I.—*Zero stands for a quantity numerically less than any finite quantity.*

For, however small a finite quantity may be, it is more than no quantity, or nothing.

336. II.—*Infinity stands for a quantity numerically greater than any finite quantity.*

$$\text{For, let } \frac{1}{x} = q.$$

Now q expresses the number of times that x is contained in 1. Hence,

1°, q is larger in proportion as x is smaller;

2°, so long as x , however small, has a finite value, q will also have a finite value.

Therefore if x be less than any finite number, or 0, q is greater than any finite number. Hence, $\frac{1}{0} = \infty$; in which ∞ stands for a quantity greater than any finite quantity.

337. III.—*A finite quantity is not changed by the addition or subtraction of zero.*

$$\text{For, since } a - a = 0,$$

$$a + 0 = a, \text{ and } a - 0 = a \dots \dots \dots (A)$$

338. IV.—*The product of zero by a finite quantity is zero.*

$$\begin{array}{ll} \text{For,} & m \times (a - a) = ma - ma \\ \text{or,} & m \times 0 = 0 \dots \dots \dots (B) \end{array}$$

339. V.—*The quotient of zero by a finite quantity is zero.*

$$\begin{array}{ll} \text{For, (B)} & 0 = m \times 0 \dots \dots \dots (1) \\ (1) \div m, & \frac{0}{m} = 0. \quad \quad \quad (C) \end{array}$$

340. VI.—*Any finite quantity divided by zero is equal to infinity.*

For, taking the reciprocal of (C),

$$\frac{m}{0} = \frac{1}{0} = \infty \dots \dots \dots (D)$$

341. VII.—*A finite quantity divided by infinity is equal to zero.*

For, multiplying (D) by $\frac{0}{\infty}$,

$$\frac{m}{\infty} = 0 \dots \dots \dots (E)$$

342. VIII.—*Infinity multiplied by a finite quantity is infinity.*

$$\begin{array}{ll} \text{For,} & m \times \frac{1}{0} = \frac{m}{0} \\ \text{or,} & m \times \infty = \infty \dots \dots \dots (F) \end{array}$$

343. IX—*Infinity is not changed by the addition or subtraction of a finite quantity.*

For in (A) substitute $\frac{m}{\infty}$ for 0 (E),

$$a + \frac{m}{\infty} = a, \text{ and } a - \frac{m}{\infty} = a.$$

Multiply by ∞ ,

$$a \times \infty + m = a \times \infty, \text{ and } a \times \infty - m = a \times \infty.$$

But $a \times \infty = \infty$ (F); hence,

$$\infty + m = \infty, \text{ and } \infty - m = \infty. \quad (G)$$

344. X.—*Each of the following may have any finite value, viz.:* $\frac{0}{0}$, $0 \times \infty$, $\frac{\infty}{\infty}$, $\infty - \infty$.

For,

$$1^{\circ}, \text{ Since (B) } m \times 0 = 0, \quad \frac{0}{0} = m \dots \dots \dots (H)$$

$$2^{\circ}, \text{ Since (D) } \frac{m}{0} = \infty, \quad 0 \times \infty = m \dots (K)$$

$$3^{\circ}, \text{ Since (F) } m \times \infty = \infty, \quad \frac{\infty}{\infty} = m \dots \dots \dots (L)$$

$$4^{\circ}, \text{ Since (G) } \infty = \infty + m, \quad \infty - \infty = m \dots (M)$$

GEOMETRICAL ILLUSTRATIONS.

345. These illustrations are intended to give the student a clearer idea of the meaning of a few of the preceding formulas.

If we denote the length of a line by a , then since a point forms no appreciable part of a line, the length of a point is 0. The meaning then,

1°. Of $a + 0 = a$, and $a - 0 = a$, is this: *a point, when added to or subtracted from a line, does not change its value.*

2°. Of $\frac{a}{0} = \infty$, *a point is contained in a line an infinite number of times.*

3°. Of $\infty \times 0 = a$, *an infinite number of points make up a line.*

4°. Of $\frac{a}{\infty} = 0$, *if a line be divided into an infinite number of parts, each part is a point.*

CHAPTER X.

RATIO, PROPORTION AND VARIATION.

RATIO.

346. Ratio is the relation between two like quantities expressed by their quotient

Thus, the ratio of m to n is $\frac{m}{n}$, which is often written $m : n$.

When the colon is thus employed it is called the sign of ratio, and is equivalent to $\div$.

If A and B are lines whose lengths are 5 and 2 inches, respectively, then

$$A : B = \frac{5 \text{ in.}}{2 \text{ in.}} = 2\frac{1}{2};$$

which is read, the ratio of A to B is $2\frac{1}{2}$.

347. Commensurable Quantities are those which contain a common unit an exact number of times.

Thus, if the lengths of the lines A and B are $1\frac{3}{4}$ and $2\frac{1}{2}$ feet, A and B are commensurable.

For, $A = 1\frac{3}{4} \text{ ft.} = 1\frac{3}{2} \text{ ft.} = 21 \text{ twelfths ft.};$

and, $B = 2\frac{1}{2} \text{ ft.} = 2\frac{1}{2} \text{ ft.} = 28 \text{ twelfths ft.}$

In this case, $A : B = \frac{21}{28} = \frac{3}{4}$.

348. Incommensurable Quantities are those which do not contain a common unit an exact number of times.

Thus, if the side of a square (S) is 1 foot, the length of the diagonal (D) is $\sqrt{2}$ feet. Now $\sqrt{2}$ and 1 are incommensurable, as $\sqrt{2}$ can not be expressed by any rational part of 1.

In this case, $D : S = \sqrt{2}$, which is called an incommensurable ratio.

349. The ratio of two incommensurable quantities may always be expressed as near the true value as we please by means of a fraction, if we only make the common unit of measure sufficiently small.

Thus (348), $\frac{1}{10}$ of a ft. is contained in S 10 times, and in D a small fraction over 14 times. Hence, we have very nearly $D : S = \frac{14}{10} = 1.4$.

Again, making the unit smaller, $\frac{1}{100}$ of a ft. is contained in S 100 times, and in D a little over 141 times. This gives a nearer approximation, viz.: $D : S = \frac{141}{100} = 1.41$.

350. The **Terms** of a ratio are the quantities compared; the **Antecedent** is the first term or dividend; the **Consequent** is the second term or divisor, and the two terms together form a **Couplet**.

351. A **Simple Ratio** is one whose terms are entire.

352. A **Compound Ratio** is the ratio of the products of the corresponding terms of two or more ratios.

353. A **Duplicate Ratio** of two quantities is the ratio of their squares.

Thus, $x^2 : y^2$ is the duplicate ratio of $x : y$.

354. The **Triplicate Ratio** of two quantities is the ratio of their cubes.

Thus, $x^3 : y^3$ is the triplicate ratio of $x : y$.

355. The **Reciprocal** of a ratio is the result of interchanging the places of its terms.

Thus, the reciprocal of $m : n$ is $n : m$.

PROPERTIES OF RATIOS.

356. I.—*If both terms of a ratio be multiplied or divided by the same number, the ratio is not altered.*

$$\text{For, } \frac{a}{b} = \frac{a \times n}{b \times n} = \frac{a \div n}{b \div n},$$

$$\text{Or, } a : b = a \times n : b \times n = a \div n : b \div n.$$

357. II.—*If both terms of a ratio be increased by the same quantity, the ratio will be made more nearly equal to unity; that is, if the ratio is a proper fraction, it will be increased; and if it is an improper fraction, it will be decreased.*

Let $a : b$, or $\frac{a}{b}$, be the ratio.

Add n to both terms, $a + n : b + n$, or $\frac{a + n}{b + n}$.

The difference between :

$$1^{\circ}. \quad 1 \text{ and } \frac{a}{b} \text{ is } 1 - \frac{a}{b} = \frac{b - a}{b}$$

$$2^{\circ}. \quad 1 \text{ and } \frac{a + n}{b + n} \text{ is } 1 - \frac{a + n}{b + n} = \frac{b - a}{b + n}.$$

$$\text{Now, numerically, } \frac{b - a}{b} > \frac{b - a}{b + n}.$$

Hence, $\frac{a + n}{b + n}$ is more nearly equal to unity than $\frac{a}{b}$, for the difference between the latter and 1 is greater than the difference between the former and 1.

EXERCISE LXXVI.

1. Compound $3 : 4$ with the duplicate of $2 : 5$.

$$\text{The duplicate of } 2 : 5 \text{ is } \frac{2^2}{5^2} = \frac{4}{25}; \quad 3 : 4 = \frac{3}{4};$$

$$\frac{4}{25} \times \frac{3}{4} = \frac{3}{25}, \text{ Ans.}$$

2. Which is the greater, the reciprocal of $5 : 2$ or the triplicate of $2 : 3$.

$$\text{The former} = \frac{2}{5}, \text{ the latter} = \frac{2^3}{3^3} = \frac{8}{27}; \quad \frac{2}{5} = \frac{54}{135}$$

$$\text{and } \frac{8}{27} = \frac{40}{135}. \text{ Hence, the former is the greater.}$$

3. Which is the greater $3 : 4$ or $5 : 6$?
4. How much less is $3 : 4$ than $(3 + 1) : (4 + 1)$?
5. How much greater is $6 : 5$ than $(6 + 2) : (5 + 2)$?
6. If $x : 3$ added $x : 4 = 14$, how much is x ?
7. If $2x : 5$ is 35 more than $3x : 11$, what is the value of x ?

Find the value of x in the following equations:

8. $x : 3$, + the reciprocal of $x : 3$, $= 3\frac{1}{3}$.
9. $x : 5$, + the duplicate of $x : 5$, $= \frac{14}{5}$.
10. The duplicate of $3x : 4 =$ the triplicate of $x : 2$.
11. Two numbers are in the ratio $3 : 4$, and if 4 be added to each, they are in the ratio $4 : 5$. Find the numbers:

Suggestion. Let $3x$ and $4x$ represent the numbers.

12. Two numbers are in the ratio $5 : 6$, and if 5 be added to each, they are in the ratio $11 : 13$. Find the numbers.

13. The ratio of the ages of A and B is $3 : 7$, and 15 years ago it was $1 : 4$. What are their ages?

14. If 3 pounds of tea are worth as much as 5 pounds of coffee, and 7 pounds of coffee as much as 11 pounds of sugar, find the ratio of the value of tea to that of sugar.

15. One-seventh of the diameter of a circle is contained nearly 22 times in the circumference. Find the approximate ratio of the circumference to the diameter of a circle.

16. Two numbers are in the ratio of 2 to 3, and if 1 be subtracted from 7 times the first and 6 be added to

5 times the second, the results are in the ratio of 5 to 6. Find the numbers.

17. There are two roads from A to B, one of them 14 miles longer than the other, and two roads from B to C, one of them 8 miles longer than the other. The distances from A to B and from B to C along the shorter roads are in the ratio of 1 to 2, and the distances along the longer roads are in the ratio of 2 to 3. Determine the distances.

PROPORTION.

358. A **Proportion** is an equality of ratios.

359. The equality of two ratios may be indicated by the sign $=$, or by the sign $::$.

The equality of $12 : 3$ and $8 : 2$ may be expressed thus,

$$12 : 3 = 8 : 2, \text{ or } 12 : 3 :: 8 : 2.$$

360. The **Proportionals** are the four numbers compared; the **Extremes** are the first and fourth terms; and the **Means** are the second and third terms.

361. The algebraic test of a proportion is that the two fractions which represent the ratios shall be equal.

362. If three quantities a, b, c , are so related that $a : b = b : c$, b is said to be a **Mean Proportional** between a and c .

363. If four quantities a, b, c, d are so related that

$$a : b = \frac{1}{c} : \frac{1}{d},$$

they are said to be **Inversely** or **Reciprocally** proportional.

364. A Continued Proportion is several equal ratios connected by the sign $=$.

$$\text{As,} \quad 4 : 2 = 6 : 3 = 14 : 7 = 22 : 11.$$

PROPERTIES OF PROPORTIONALS.

365. I.—*The product of the extremes is equal to the product of the means.*

$$\text{For, if} \quad a : b :: c : d \dots\dots (1)$$

$$\text{Then} \quad \frac{a}{b} = \frac{c}{d} \dots\dots\dots (2)$$

$$(2) \times bd, \quad ad = bc.$$

366. II.—*If the product of two numbers be equal to the product of two others, the four numbers are proportionals.*

$$\text{Let} \quad ad = bc \dots\dots\dots (1)$$

$$(1) \div bd \quad \frac{a}{b} = \frac{c}{d} \dots\dots\dots (2)$$

$$\text{Or} \quad a : b :: c : d.$$

367. III.—*Four proportionals are in proportion:*

1^o, *By Inversion.* That is,

The second : the first :: the fourth : the third.

$$\text{Let} \quad a : b :: c : d.$$

$$\text{Then (365),} \quad bc = ad \dots\dots\dots (1)$$

$$(1) \div ac, \quad \frac{b}{a} = \frac{d}{c},$$

$$\text{Or,} \quad b : a :: c : d.$$

368. 2^o, By Alternation. That is,

The first : the third :: the second : the fourth.

$$\text{Let} \quad a : b :: c : d.$$

$$\text{Then} \quad ad = bc \dots\dots\dots (1)$$

$$(1) \div cd, \quad \frac{a}{c} = \frac{b}{d},$$

$$\text{Or} \quad a : c :: b : d.$$

369. 3^o, By Composition. That is, .

The first + the second : the second :: the third + the fourth : the fourth.

$$\begin{array}{ll}
 \text{Let} & a : b :: c : d \\
 \text{then} & \frac{a}{b} = \frac{c}{d} \dots\dots\dots (1) \\
 \text{Evidently,} & \frac{b}{b} = \frac{d}{d} \dots\dots\dots (2) \\
 (1) + (2), & \frac{a+b}{b} = \frac{c+d}{d} \dots\dots\dots (3) \\
 \text{or,} & a+b : b :: c+d : d.
 \end{array}$$

370. 4^o, By Division. That is,

The first — the second : the second :: the third — the fourth : the fourth

$$\begin{array}{ll}
 \text{Let} & a : b :: c : d \\
 \text{then} & \frac{a}{b} = \frac{c}{d} \dots\dots\dots (1) \\
 \text{Evidently,} & \frac{b}{b} = \frac{d}{d} \dots\dots\dots (2) \\
 (1) - (2), & \frac{a-b}{b} = \frac{c-d}{d} \dots\dots\dots (3) \\
 \text{or,} & a-b : b :: c-d : d.
 \end{array}$$

371. 5^o, By Composition and Division.

For, dividing (3) of Art. 369 by (3) of Art. 370, we have

$$\begin{array}{l}
 \frac{a+b}{a-b} = \frac{c+d}{c-d} \\
 \text{Or,} \quad a+b : a-b :: c+d : c-d.
 \end{array}$$

372. IV.—A mean proportional between two quantities is equal to the square root of their product.

$$\begin{array}{ll}
 \text{Let} & a : b :: b : c \\
 \text{then} & b^2 = ac. \\
 \text{Whence,} & b = \sqrt{ac}.
 \end{array}$$

373. V.—*The products of the corresponding terms of two or more proportions are in proportion.*

$$\text{Let } a : b :: c : d, \text{ then } \frac{a}{b} = \frac{c}{d} \dots\dots\dots (1)$$

$$\text{Let } e : f :: g : h, \text{ then } \frac{e}{f} = \frac{g}{h} \dots\dots\dots (2)$$

$$\text{Let } m : n :: r : s, \text{ then } \frac{m}{n} = \frac{r}{s} \dots\dots\dots (3)$$

$$(1) \times (2) \times (3), \quad \frac{aem}{bfns} = \frac{cgr}{dhs}$$

$$\text{Or,} \quad aem : bfn :: cgr : dhs.$$

374. VI.—*Like powers, or like roots, of proportionals are in proportion.*

$$\text{Let } a : b :: c : d, \text{ then } \frac{a}{b} = \frac{c}{d} \dots\dots\dots (1)$$

$$(1)^n, \quad \frac{a^n}{b^n} = \frac{c^n}{d^n}, \text{ or } a^n : b^n :: c^n : d^n.$$

$$\sqrt[n]{(1)}, \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{\sqrt[n]{c}}{\sqrt[n]{d}}, \text{ or } \sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}.$$

375. VII.—*In any continued proportion, any antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents.*

$$\text{Let } a : b = c : d = e : f,$$

$$\text{Then } ad = bc \dots\dots\dots (1)$$

$$\text{And } af = be \dots\dots\dots (2)$$

$$\text{Also, } ab = ba \dots\dots\dots (3)$$

$$(1) + (2) + (3), + a(b + d + f) = b(a + c + e)$$

$$\text{Whence (366), } a : b :: a + c + e : b + d + f.$$

EXERCISE LXXVII.

If $a : b :: c : d$, prove that

$$1. a^2 : b^2 :: c^2 : d^2.$$

$$2. an : bn :: cr : dr.$$

$$3. an : br :: cn : dr.$$

$$4. \frac{1}{2}a : \frac{1}{2}b :: \frac{1}{3}c : \frac{1}{3}d.$$

$$5. 2a + b : 2b :: 2c + d : 2d.$$

6. $3a + 2b : 2a - 3b :: 3c + 2d : 2c - 3d.$

7. $a = \frac{bc}{d}; b = \frac{ad}{c}; c = \frac{ad}{b}; d = \frac{bc}{a}.$

8. Find a mean proportional between 6 and 24.

9. Find x when $6 : 15 :: 14 : x.$

10. Find x when $\frac{1}{2} : x :: \frac{1}{3} : 3\frac{1}{5}.$

11. Find x when $x + 2 : x - 3 :: 3 : 2.$

12. Find x when $x + 5 : 2x - 3 :: 5x + 1 : 3x - 3.$

13. Find x when $\sqrt{x} + \sqrt{c} : \sqrt{x} - \sqrt{c} :: a : c.$

14. Find x when $\frac{x^2 + x + 1}{x^2 - x + 1} = \frac{2x + 1}{2x - 1}.$

15. Two numbers are in the ratio of 2 to 3, and if 9 be added to each they are in the ratio of 3 to 4. Find the numbers.

16. There are two numbers, whose sum is to their difference as 8 to 1, and the difference of whose squares is 128. What are the numbers?

17. Of two square grass-plots, a side of one is 10 yards longer than the side of the other, and their areas are as 25 to 9. Find the lengths of the sides.

18. There are two numbers, whose sum is to the less as 5 to 2, and whose difference is to 9 as 15 is to the difference of their squares. Find the numbers.

19. There are two numbers, whose sum is 10, and the difference of whose fourth powers contains the sum of their squares as often as 26 contains 7. Find the numbers.

20. Divide the number 49 into two such parts, that the ratio of the first to the second may be to the ratio of the second to the first as $\frac{4}{3}$ to $\frac{3}{4}.$

21. If $a : b :: c : d$ and $c : b :: b : d$, prove that $a : c :: c : b$.

22. Of four proportionals of the same kind, show that the sum of the greatest and least is greater than the sum of the other two.

NOTE I.—The ratio of the areas of two circles is the duplicate ratio of their diameters.

23. The ratio of the diameters of two circular gardens is 5 : 8; what will it cost to mow the larger if it cost \$10 to mow the smaller?

24. The ratio of the areas of two circles, which is 1 : 4, becomes 4 : 9 when each of the diameters is increased by 2. Find the diameters.

NOTE II.—The ratio of the volumes of two spheres is the triplicate ratio of their diameters.

25. The ratio of the diameters of two balls is 4 to 7; what is the weight of the larger if the smaller weigh 32 pounds?

26. The ratio of the volumes of two spheres, which is 1 : 8, becomes 8 : 27 when each of the diameters is increased by 3. Find the diameters.

VARIATION.

376. A Constant is a quantity that is regarded as having a fixed value.

Constants are generally represented by figures or by the first letters of the alphabet.

377. A Variable is a quantity that is, or is conceived to be, continually changing in value.

Variables are generally represented by the last letters of the alphabet.

378. A **Function** of a variable is a quantity or algebraic expression whose value depends on the variable.

Thus, u is a function:

1°, Of the variable x in $u = 3x^2 - 2x + 5$;

2°, Of the variable y in $u = y^3 - ay^2 + b$.

In these functions 3, 2, 5, a and b are constants.

379. A quantity may be a function of two or more variables.

Thus, u is a function:

1°, Of x and y in $u = x^2 - 2xy + 7y$;

2°, Of x , y and z in $u = x^2y - cxz^2 + exyz$.

380. One quantity *varies* directly as another when the two quantities are such functions of each other that if the one be changed the other is changed in the same proportion.

Thus, the space (s) described by a body moving with a uniform velocity (v) varies directly as the time (t).

Sometimes, for brevity, we omit the word *directly*, and say simply that one quantity varies as another.

381. The **Sign of Variation** is $\propto$.

Thus, $s \propto t$ is read s varies as t .

When the word *let* precedes an expression like $s \propto t$, the symbol $\propto$ is read *vary as*.

382. One quantity varies *inversely* as another when the first varies as the reciprocal of the second.

Thus, in $a = xy$, $x \propto \frac{1}{y}$.

383. One quantity varies as two others *jointly* when the first varies as the product of the others.

Thus, if the area of a triangle is y , its base x and its altitude z , then y varies as x and z jointly, or $y \propto xz$.

384. One quantity varies directly as a second quantity and inversely as a third, when the first varies as the quotient obtained by dividing the second by the third,

Thus, in uniform motion, the time (t) varies directly as the space (s) and inversely as the velocity (v); that is $t \propto \frac{s}{v}$.

PRINCIPLES.

385. I.—*If one quantity varies as another, either of them is equal to a constant multiple of the other.*

Let $x \propto y$.
and suppose $y = b$ when $x = a$; then (380)
 $x : a = y : b$.

Whence, $y = \frac{b}{a} x = my$;

where m is equal to the constant $\frac{b}{a}$.

386. II.—*If one variable is equal to a constant multiple of another, the first varies as the second.*

Let $y = mx$,
and suppose $y = b$ when $x = a$; then
 $b = ma$;

$\therefore y : b = x : a$;

That is, $y \propto x$.

387. III.—*If one quantity varies as a second, and the second as a third, the third varies as the first.*

Let $x \propto y$ and $y \propto z$,
Then $y = mx$ and $z = ny$;
Whence, $z = mn x$, or $z \propto x$.

388. IV.—*If each of two quantities varies as a third, their sum, their difference, or their mean proportional varies as the third.*

Let $x \propto z$ and $y \propto z$,
Then $x = mz$ and $y = nz$; whence
 $x + y = (m + n)z$, $x - y = (m - n)z$, and $\sqrt{xy} = (\sqrt{mn})z$.
Or, $x + y \propto z$, $x - y \propto z$, and $\sqrt{xy} \propto z$,

389. V.—If one quantity varies as two others jointly, either of the latter varies directly as the first and inversely as the other.

Let $x \propto yz$, then $x = myz$;

Whence, $y = \left(\frac{1}{m}\right)\frac{x}{z}$, and $z = \left(\frac{1}{m}\right)\frac{x}{y}$,

$$y \propto \frac{x}{z}, \text{ and } z \propto \frac{x}{y}.$$

NOTE.—Several other important principles are given in the following exercises:

EXERCISE LXXVIII.

1. If $x \propto y$, prove that $xz \propto yz$.

Since $x \propto y$, $x = my \dots \dots \dots (1)$

$$(1) \times z, \quad xz = myz; \quad \therefore xz \propto yz.$$

2. If $x \propto y$, prove that $\frac{x}{z} \propto \frac{y}{z}$.

3. If $x \propto y$, and $u \propto v$, prove that $ux \propto vy$, also $\frac{u}{x} \propto \frac{v}{y}$.

4. If $x \propto y$, prove that $x^n \propto y^n$.

5. If $x \propto z$ and $y \propto z$, prove that $\sqrt{x^2 - y^2} \propto z$.

6. If $x \propto y$ and $x = 2$ when $y = 10$, what is the value of y in terms of x .

Since $x \propto y$, $y = mx \dots \dots \dots (1)$

In (1) substitute 2 for x and 10 for y ,

$$10 = 2m; \text{ whence, } m = 5.$$

Now substitute 5 for m in (1), $y = 5x$, Ans.

7. If $x \propto y$ and $x = 4$ when $y = 6$, find the value of y in terms of x .

8. If $x \propto y$ and $x = 3$ when $y = 12$, find the value of y when $x = 5$.

9. If $z \propto x + y$ and $z = 15$ when $x = 3$ and $y = 2$, find the value of z in terms of x and y .

10. If $x^2 \propto y^3$ and $x = 4$ when $y = 2$, find the value of y in terms of x .

11. If $x \propto y^2$ and $z \propto y^3$ and $z = 6$ when $y = 2$ and $x = 1$, find the value of z in terms of xy .

12. If $x \propto y$ when z is constant, and if $x \propto z$ when y is constant, how does x vary when both y and z are variable?

The variation of x depends upon the variations of the two quantities y and z ; let the variations of the latter quantities take place separately, and when y is changed to b let x be changed to a ; then

$$\frac{x}{a} = \frac{y}{b} \dots \dots \dots (1)$$

Now let z be changed to c , and in consequence let x be changed from a to n ; then

$$\frac{a}{n} = \frac{z}{c} \dots \dots \dots (2)$$

$$(1) \times (2), \quad \frac{x}{n} = \frac{yz}{bc}, \text{ or } x = \left(\frac{n}{bc}\right)yz.$$

$$\therefore \quad \quad \quad x \propto yz.$$

13. If $u \propto x$ when y and z are constant, and $u \propto y$ when x and z are constant, and $u \propto z$ when x and y are constant, how does u vary when x , y and z are variable.

14. If one quantity vary directly as another, and the former be $\frac{3}{4}$ when the latter is $\frac{4}{3}$, what will the latter be when the former is 9?

15. Given that the area of a circle varies as the square of its radius, prove that the area of a circle whose radius is 5 inches is equal to the sum of the areas of two circles whose radii are 3 and 4 inches.

16. Given that the volume of a sphere varies as the cube of its radius, prove that the volume of a sphere whose radius is 6 inches is equal to the sum of the volumes of three spheres whose radii are 3, 4 and 5 inches.

Application to Problems in Proportion.

17. If 4 hats cost \$15, how much will 7 hats cost?

Suggestion. Let x denote hats and y the value of hats, then $x \propto y$, whence $x = my$.

18. If \$40 draw \$8 interest in a certain time, what interest will \$90 draw in the same time?

19. If a pole m ft. high cast a shadow n ft. long, what is the height of a tree that casts a shadow r ft. long?

20. If 4 men in 9 days can do a certain work, how long will it take 10 men to do the same work?

Suggestion. Let x denote men and y time, then $x \propto \frac{1}{y}$

whence $x = \frac{m}{y}$.

21. If 25 men can do a piece of work in 24 days, how many men can do the same in 30 days?

22. If 15 persons consume 6 barrels of flour in 9 months, how long will it take 20 persons to consume $10\frac{2}{3}$ barrels?

Let x denote number of persons, y time and z number of barrels of flour; then $x \propto \frac{z}{y}$; that is $x \propto z$ when y is constant, and x varies inversely as y when z is constant.

$$\text{Hence, } x = \frac{mz}{y} \dots \dots \dots (1)$$

Now when $x = 15$, $y = 9$ and $z = 6$. Substituting in (1) we find $m = \frac{45}{2}$. Therefore, $x = \frac{45z}{2y} \dots \dots \dots (2)$

Now we wish to know the value of y when $x = 20$ and $z = 10\frac{2}{3}$. Making these substitutions in (2) we find $y = 12$, Ans.

23. If 6 persons consume 240 pounds of sugar in 8 months, how much will 9 persons consume in 11 months?

24. If 12 men can do as much work as 25 women, and 5 women do as much as 6 boys, how many men would it take to do the work of 75 boys?

25. If 12 oxen eat $3\frac{1}{3}$ acres of grass in 4 weeks, and if 21 oxen eat 10 acres in 9 weeks, how many oxen will eat 24 acres in 18 weeks, the grass growing uniformly?

Let x denote oxen, t weeks, y acres of grass, z growth of grass.

1°. The amount of growth varies as the number of acres multiplied by the time. Hence,

$$z \propto ty, \text{ or } z = nty \dots \dots \dots (1)$$

2°. The number of oxen varies as the total amount of grass divided by the time. Hence,

$$x \propto \frac{y+z}{t}, \text{ or } tx = my + mz \dots \dots \dots (2)$$

Combine (1) and (2), making $mn = r$,

$$tx = my + rty \dots \dots \dots (3)$$

Now, when $x = 12$, $y = 3\frac{1}{3}$ and $t = 4$. Hence,

$$10m + 40r = 144 \dots \dots \dots (4)$$

Again, when $x = 21$, $y = 10$ and $t = 9$. Hence,

$$10m + 90r = 189 \dots \dots \dots (5)$$

Combining (4) and (5), $r = \frac{9}{10}$, $m = \frac{108}{10}$. Substitute in (3), reduce,

$$10tx = 108y + 9ty \dots \dots \dots (6)$$

Now, in (6) make $t = 18$, $y = 24$, and we get $x = 36$; hence, 36 oxen. Ans.

26. A cistern has a stream of water running into it; it has 10 pipes; all together will empty it in $2\frac{1}{2}$ hr., and 6 pipes will empty it in $5\frac{1}{2}$ hr. In what time will 3 pipes empty it?

CHAPTER XI.

SERIES.

390. A **Series** is a succession of quantities, so related that each may be derived from one or more preceding ones, in accordance with some fixed law.

391. The **Terms** of a series are the quantities of which it is formed:

392. The **Extremes** of a series are the first and last terms, and the **Means** are the terms between the extremes.

ARITHMETICAL PROGRESSION.

393. An **Arithmetical Progression** is a series in which each term may be derived from the preceding term by the addition of a *common difference*.

394. An **Increasing Arithmetical Progression** is one in which the common difference is positive.

Thus, 3, 7, 11, 15, 19, 23, etc., is an increasing arithmetical progression in which the common difference is $+4$.

395. A **Decreasing Arithmetical Progression** is one in which the common difference is negative.

Thus, 13, 8, 3, -2 , -7 , -12 , etc., is a decreasing arithmetical progression in which the common difference is -5 .

396. There are five parts or elements in an arithmetical progression, viz.: *first* term, *last* term, *number* of terms, *common difference*, and *sum* of all the terms.

These parts are so related that any two of them may be found when the other three are given.

397. *To find the last term when the first term, the common difference and the number of terms are given.*

Let a denote the first term, d the common difference, n the number of terms, and l the last term; then the progression will be

$$a, (a + d), (a + 2d), (a + 3d), (a + 4d), \text{ etc.}$$

That is, the coefficient of d in any term is one less than the number of that term in the series; consequently, the n th or last term will be

$$a + (n - 1)d.$$

Whence, putting l for the n th term, we have

$$l = a + (n - 1)d,$$

in which d is either positive or negative, according as the series is an increasing or a decreasing one.

Hence, Rule. *To the first term add the product of the common difference by the number of terms less one.*

EXERCISE LXXIX.

1. If the first term is 4, the common difference 5, and the number of terms 30, what is the last term?

2. If the first term is 7, the common difference 3, and the number of terms 20, what is the last term?

3. What is the tenth term if the first term is 23 and the common difference -2 ?

4. When the first term is 31 and the common difference -3 , what is the fifteenth term?

5. When the first term is $5\frac{1}{3}$ and the common difference $3\frac{1}{3}$, what is the twelfth term?

6. Find the 13th term of the series 11, 16, 21, etc.

7. What is the 21st term of the series 103, $100\frac{1}{2}$, 98, $95\frac{1}{2}$, etc.?

398. *To find the sum of the terms when the first term, common difference and number of terms are given.*

Let a denote the first term, d the common difference, n the number of terms, l the last term, and S the sum of the terms. Then,

$$s = a + (a + d) + (a + 2d) + \dots + l,$$

or, writing the terms in the reverse order,

$$s = l + (l - d) + (l - 2d) + \dots + a.$$

Therefore, by adding these equations, term by term,

$$2s = (a + l) + (a + l) + (a + l) + \dots + (a + l).$$

Here $(a + l)$ is taken as many times as there are terms, or n times; whence,

$$2s = n(a + l), \text{ or } s = \frac{1}{2}n(a + l).$$

Hence, Rule I.—*Find the last term (397).*

II.—*Multiply the sum of the first and last terms by half the number of terms.*

8. If the first term is 5, the last term 62, and the number of terms 20, what is the sum of the terms?

9. If the least term of a series of numbers in arithmetical progression be 4, the greatest 100, and the number of terms 17, what is the sum of the terms?

10. Find the sum of the natural series of numbers 1, 2, 3, 4, etc., carried to 1000 terms.

11. Find the sum of n terms of the same series.

12. Required the sum of the series of odd numbers 1, 3, 5, 7, etc., carried to 101 terms.

13. Find the sum of n terms of the same series.

14. How many times does the hammer of a common clock strike in a week?

Find the sum

15. Of 20 terms of the series 2, 6, 10, 14, etc.
16. Of 32 terms of the series 4, $3\frac{3}{4}$, $3\frac{1}{2}$, $3\frac{1}{4}$, etc.
17. Of 12 terms of the series 1, $1\frac{3}{4}$, $2\frac{1}{2}$, etc.
18. Of 50 terms of the series $\frac{1}{3}$, $\frac{2}{3}$, 1, etc.
19. Of n terms of the series 9, 11, 13, etc.
20. Of n terms of the series 1, $\frac{5}{6}$, $\frac{2}{3}$, etc.
21. Of 30 terms of the series -27 , -20 , -13 , etc.
22. Of n terms of the series $\frac{1}{2}$, $-\frac{2}{3}$, $-\frac{11}{6}$, etc.
23. Of n terms of the series $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, etc.

399. *To insert a given number of arithmetical means between two given terms.*

Let it be required to insert m arithmetical means between a and l .

Since there are to be m terms between a and l , there will be $m + 2$ terms in all, a being the first and l the last. Now to find d , the common difference, substitute $m + 2$ for n in the formula $l = a + (n - 1)d$, which gives, after reducing, $d = \frac{l - a}{m + 1}$.

Hence, to find the common difference, *divide the difference of the two numbers by the number of means plus one.*

24. Insert 4 arithmetical means between 3 and 18.

$$\text{Common difference} = \frac{18 - 3}{4 + 1} = 3,$$

Hence, the means are $3 + 3 = 6$, $6 + 3 = 9$, $9 + 3 = 12$,
 $12 + 3 = 15$.

25. Insert 1 arithmetical mean between 5 and 13.
26. Insert 5 arithmetical means between 2 and 26.
27. Insert 7 arithmetical means between 5 and 33.
28. Insert 9 arithmetical means between 3 and 4.
29. Insert 15 arithmetical means between $4\frac{1}{6}$ and $5\frac{1}{2}$.

400. The formulas $l = a + (n - 1)d \dots\dots (A)$

$$s = \frac{1}{2} n(a + l) \dots\dots\dots (B)$$

form two independent equations, and contain together all the five elements of an arithmetical progression. Hence, when either three of these elements are given, the other two may be determined either from one of the equations or from combining the equations.

30. Required the first term of an arithmetical progression, when the last term is 62, the common difference 3, and the number of terms 20.

The required term (a) and the three given terms (l , d and n) are all contained in (A); hence, the problem may be solved by substituting 62 for l , 3 for d and 20 for n in (A), and reducing. Thus,

$$62 = a + (20 - 1)3; \text{ whence, } a = 5.$$

31. Find the common difference, when the last term is 149, the first term 4, and the number of terms 30.

32. Find the number of terms, when the last term is 59, the first term 3, and the common difference 4.

33. Find the first term, when the sum of the terms is 670, the number of terms 20, and the last term 62.

34. When the first term is 7, the number of terms 6, and the sum of the terms — 18; what is the last term?

35. The extremes are 2 and 52, and the sum of the terms 297; what is the number of terms?

36. When the last term is 1, the first term — 6, and the number of terms 15; what is the common difference?

37. The extremes are 3 and 45, and the common difference 2; what is the sum of the series?

Since neither (A) nor (B) contains all of the given parts and the required part, it is necessary to use both formulas.

Substituting 3 for a , 45 for l , and 2 for d in (A) and (B), we obtain

$$45 = 3 + 2(n - 1) \dots \dots \dots (1)$$

$$S = \frac{1}{2}n(3 + 45) \dots \dots \dots (2)$$

Now, combining (1) and (2) so as to eliminate the superfluous element n , we obtain $S = 528$, Ans.

38. The sum of a certain number of terms of the series 19, 17, 15, etc., is 91; find the number of terms?

Here, $a = 19$, $d = -2$, and $S = 91$. Substituting in (A) and (B), we have

$$l = 19 - 2(n - 1) \dots \dots \dots (1)$$

$$91 = \frac{1}{2}n(19 + l) \dots \dots \dots (2)$$

Combining (1) and (2) so as to eliminate the superfluous element l , we have

$$n^2 - 20n = -91.$$

Whence, $n = 13$, or 7.

39. What is the common difference when the first term is 1, the last 50, and the sum of the terms 204.

40. The first term of a decreasing arithmetical series is 10, the number of terms 10, and the sum of the terms 85; find the last term and the common difference.

41. The last term of an arithmetical progression is 52, the common difference 5, and the sum of the series 297; find the first term and the number of terms.

42. Insert 4 arithmetical means between 5 and 7.

43. The sum of the first two terms of an arithmetical progression is 4, and the fifth term is 9; find the series.

44. What is the series of which $\frac{1}{2}n(n - 1)$ is the n th term?

Substitute 1, 2, 3, 4, etc., successively for n .

45. The n th term of an arithmetical progression is $\frac{1}{2}(3n - 1)$; find the first term, the common difference, and the sum of n terms.

46. How many terms of $16 + 24 + 32 + 40 + \text{etc.}$ amount to 1840?

47. How many terms of the series 5, 4, 3, etc., must be taken to make 14?

48. Find the sum of n terms of $1 - 3 + 5 - 7 + \text{etc.}$

49. Find the sum of n terms of $1 - 2 + 3 - 4 + \text{etc.}$

50. Find four numbers in arithmetical progression, such that the product of the extremes shall be 27, of the means 35.

Let $x - 3y, x - y, x + y, x + 3y$, be the numbers.

51. There are 4 numbers in arithmetical progression, their sum is 32, and their product 3465; find the numbers.

52. The sum of 3 numbers in arithmetical progression is 30, and the sum of their squares 308; find the numbers.

Let $x - y, x, x + y$, be the numbers.

53. The sum of the 4th powers of three consecutive numbers is 353; find the numbers.

54. The product of 4 consecutive numbers is 1680; find the numbers.

55. The sum of 7 numbers in arithmetical progression is 35, and the sum of their cubes 1295; find the numbers.

56. Two travelers (A and B) set out from the same place at the same time; A travels uniformly at the rate

of 3 miles an hour, but B's rate is 4 miles the first hour, $3\frac{1}{2}$ the second, 3 the third, and so on. In how many hours will A overtake B?

57. The sum of 6 terms of an arithmetical progression is 93, and the 6th term is 28; what is the series?

58. A traveler (A) sets out for a certain place, and goes 1 mile the first day, 2 the second, 3 the third, and so on. In 5 days afterward another (B) sets out, and travels 12 miles a day. How long and how far must B travel before he and A will be together?

59. If a falling body descends $16\frac{1}{2}$ feet the first second, three times this distance the next, five times the next, and so on, how far will it fall the 30th second, and how far altogether in half a minute?

60. Two hundred stones being placed on the ground in a straight line, at the distance of 2 feet from each other; how far will a person travel who shall bring them separately to a basket, which is placed 20 yards from the first stone, if he starts from the spot where the basket stands?

61. Prove that the sum of any $2n + 1$ consecutive integers is divisible by $2n + 1$.

GEOMETRICAL PROGRESSION.

401. A **Geometrical Progression** is a series in which each term may be derived from the preceding term by multiplying it by a *common ratio*.

402. An **Increasing Geometrical Progression** is one in which the ratio is greater than unity.

Thus, 3, 6, 12, 24, 48, etc., is an increasing geometrical progression in which the ratio is 2.

403. A Decreasing Geometrical Progression is one in which the ratio is less than unity.

Thus, 27, 9, 3, 1, $\frac{1}{3}$, etc., is a decreasing geometrical progression in which the ratio is $\frac{1}{3}$.

404. There are five parts or elements in a geometrical progression, viz.: *first* term, *last* term, *number* of terms, *ratio*, and *sum* of all the terms. These parts are so related that any two of them may be found when the other three are given.

405. To find the last term when the first term, the ratio, and the number of terms are given.

Let a denote the first term, r the ratio, and n the number of terms; then the successive terms of the series will be

$$a, ar, ar^2, ar^3, ar^4, \dots, ar^{n-1}.$$

That is, the given first term is a factor of each of the terms, and the exponent of r in the *second* term is 1, in the *third* term 2, in the *fourth* term 3, and so on, to the n th term, in which it is $n - 1$.

Therefore, if l denote the last term, we shall have

$$l = ar^{n-1}.$$

Hence, Rule.—*Multiply the first term by the ratio raised to a power whose exponent is one less than the number of terms.*

EXERCISE LXXX.

1. Find the last term of a geometrical progression whose first term is 5, ratio 3, and number of terms 7.

$$l = ar^{n-1} = 5 \times 3^{7-1} = 5 \times 3^6 = 3645.$$

2. Find the 9th term of a geometrical progression whose first term is 5, and ratio 4.

3. Find the 8th term of a geometrical progression whose first term is 28672, and ratio $\frac{1}{4}$.

4. Find the 7th term of the series 5, 10, 20, etc.

5. Find the 6th term of the series $3\frac{3}{8}, 2\frac{1}{4}, 1\frac{1}{2}$, etc.

6. Find the 8th term of the series $-21, 14, -9\frac{1}{3}$, etc.

406. *To find the sum of the terms when the first term, the ratio, and the number of terms are given.*

Let a be the first term, r the ratio, n the number of terms, and s the sum of terms. Thus

$$s = a + ar + ar^2 + \dots + ar^{n-1} \dots (1)$$

$$(1) \times r, \quad sr = ar + ar^2 + \dots + ar^{n-1} + ar^n \dots (2)$$

$$(2) - (1), \quad sr - s = ar^n - a.$$

$$\text{Whence} \quad s = \frac{a(r^n - 1)}{r - 1} \dots (3)$$

If l denote the last term, we have $l = ar^{n-1}$.

Hence, substituting in (3),

$$s = \frac{rl - a}{r - 1} \dots (4)$$

Equation (3) gives the value of s in terms of the given parts, but (4) is often a more convenient form.

Hence, Rule.—*Multiply the last term by the ratio, subtract the first term, and divide the remainder by the ratio less one.*

7. Find the sum of a geometrical series whose first term is 1, ratio 2, and last term 1024.

8. Find the sum of a geometrical series whose first term is $1\frac{1}{2}$, ratio 4, and number of terms 9.

Find the sum:

9. Of 9 terms of the series $1 + 3 + 9 + \dots$

10. Of 7 terms of the series $8 + 20 + 50 + \dots$

11. Of 6 terms of the series $1\frac{2}{5} + 2\frac{2}{3} + 4\frac{4}{9} + \dots$

12. Of 10 terms of the series $2 - 2^2 + 2^3 - \dots$

13. Of n terms of the series $3 + 2 + \frac{4}{3} + \dots$

14. Of n terms of the series $\frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \dots$

407. *To find the sum of the terms when the progression is decreasing and the number of terms infinite.*

The value of s may be written thus,

$$s = \frac{a(1 - r^n)}{1 - r}.$$

Now suppose r less than unity; then the larger n is the smaller will r^n be, and by taking n large enough r^n can be made as small as we please. If then n be taken so large that r^n may be neglected in comparison with unity, the value of s reduces to $\frac{a}{1-r}$. This is the case when n is infinitely great.

Hence, Rule.—*Divide the first term by one less the ratio.*

15. Find the sum of 4, 2, 1, $\frac{1}{2}$, etc., to infinity.

$$s = \frac{a}{1-r} = \frac{4}{1-\frac{1}{2}} = 8, \text{ Ans.}$$

16. Find the sum of 9, 3, 1, $\frac{1}{3}$, etc., to infinity.

17. Find the sum of $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{12}$, etc., to infinity.

18. Sum to infinity $5 - \frac{1}{2} + \frac{1}{20} - \frac{1}{200} + \text{etc.}$

19. Sum to infinity $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \text{etc.}$

20. Sum to infinity $\frac{3}{2} - \frac{2}{3} + \frac{8}{27} - \text{etc.}$

21. Sum to infinity $1 + \frac{1}{4} + \frac{1}{16} + \text{etc.}$

22. Sum to infinity $1 - \frac{1}{4} + \frac{1}{16} - \text{etc.}$

23. Sum to infinity $1 + \frac{1}{n} + \frac{1}{n^2} + \text{etc.}$

24. Sum to infinity $1 - \frac{1}{n} + \frac{1}{n^2} - \text{etc.}$

408. *To insert a given number of geometrical means between two terms.*

Let it be required to insert m geometrical means between a and l .

Since there are to be m terms between a and l , counting a and l there will be $m + 2$ terms in all. Now, to find r , the ratio, substitute $m + 2$ for n in the formula $l = ar^{n-1}$, and we

$$\text{obtain } r = \sqrt[m+1]{\frac{l}{a}}.$$

Hence, to find the ratio, *Divide the last term by the first, and extract that root of the quotient whose index is one more than the number of means.*

25. Insert 4 geometrical means between 7 and 224.

$$\text{Ratio} = \sqrt[4+1]{\frac{224}{7}} = \sqrt[5]{32} = 2.$$

Hence, the means are $7 \times 2 = 14$, $14 \times 2 = 28$, $28 \times 2 = 56$, $56 \times 2 = 112$.

Find:

26. The geometrical mean between 4 and 9.

27. The geometrical mean between $\frac{1}{8}$ and $\frac{5}{8}$.

28. The geometrical mean between $+1$ and -1 .

Insert:

29. Two geometrical means between 5 and 320.

30. Three geometrical means between 6 and 486.

31. Seven geometrical means between 2 and 13122.

32. Three geometrical means between $+1$ and -1 .

33. Five geometrical means between $+1$ and -1 .

409. The formulas $l = ar^{n-1}$ (C)

$$s = \frac{rl - a}{r - 1} \text{(D)}$$

form two independent equations, and contain together the five elements of a geometrical progression. Hence, in general, when either three of these elements are given the other two may be determined either from one of the equations, or from combining the equations.

34. Find the first term of a geometrical progression when the last term is 1536, the ratio 2, and the number of terms 10.

35. The last term of a geometrical progression is 12500, the first term 4, and the number of terms 6; find the ratio and sum of the terms.

36. The last term of a geometrical progression is 1536, the first term 3, and the sum of the terms 3069; find the ratio.

37. The last term of a geometrical progression is 885735, the ratio 3, and the sum of the terms 1328600; find the first term.

38. The first term of a geometrical progression is 1, the ratio 3, and the sum of the terms 9841; find the last term and the number of terms.

39. The first two terms of a series in geometrical progression are $\frac{2}{3}$ and $\frac{1}{2}$; what are the next two terms?

40. The first and third terms of a geometrical series are 4 and 16; what is the fourth term?

NOTE.—Numbers in geometrical progression may be denoted by x, xy, xy^2 , etc., in which x is the first term and y the ratio.

Again, three terms may be expressed by x^2, xy, y^2 ; four terms by $\frac{x^2}{y}, x, y, \frac{y^2}{x}$; five terms by $\frac{x^3}{y}, x^2, xy, y^2, \frac{y^3}{x}$; etc.

In all these cases the square of any term is equal to the product of the two adjacent terms.

41. The sum of three numbers in geometrical progression is 7; and the sum of their reciprocals is $\frac{7}{4}$; find them.

42. There are four numbers in geometrical progres-

sion, the sum of the first and third is 35, and the sum of the second and fourth is 70; find them.

43. There are four numbers in geometrical progression. the sum of the extremes is 28, and the sum of the means 12; find them.

44. There are three numbers in geometrical progression whose product is 27, and the sum of their cubes 757; find them.

45. There are four numbers in geometrical progression, which, being increased by 5, 7, 7, and 3 respectively, the sums are in arithmetical progression; find them.

46. There are four numbers in arithmetical progression, which, being increased by 2, 4, 8, and 15 respectively, the sums are in geometrical progression; find them.

47. Suppose a body to move 20 miles the first minute, 19 miles the second, $18\frac{1}{20}$ the third, and so on forever; required the utmost distance it can reach.

48. A father divided \$147 among three sons, the shares being in geometrical progression; and the first had \$63 more than the last. How much had each?

49. Find the geometrical series in which the sum of the first two terms is 36, and of the next two 324.

50. A person who saved every year half as much again as he saved the previous year had in 7 years saved £102 19s. How much did he save the first year?

51. There are four numbers, the first three of which are in geometrical progression, and the last three in

arithmetical progression; the sum of the first and last is 14, and the sum of the second and third is 12; find the numbers.

HARMONICAL PROGRESSION.

410. Quantities are in **Harmonical Progression** when their reciprocals form an arithmetical progression.

Thus, $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, etc.,
are in harmonical progression, because their reciprocals
 $1, 3, 5, 7$, etc.,
form an arithmetical progression.

411. *To find the relation between any three consecutive terms.*

Let a, b, c be the three terms, then their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ will be in arithmetical progression. Hence,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a} \dots\dots\dots (1)$$

$$(1) \times abc, \quad ab - ac = ac - bc$$

$$\text{or,} \quad a(b - c) = c(a - b) \dots\dots\dots (2)$$

$$(2) \div c(b - c), \quad \frac{a}{c} = \frac{a - b}{b - c}$$

Hence, of three consecutive terms of a harmonical progression, *the first is to the third as the first minus the second is to the second minus the third.*

EXERCISE LXXXI.

1. The first two terms of a harmonical progression are 30 and 10; find the third term.

Let x = the third term,

$$\text{then } \frac{30}{x} = \frac{30 - 10}{10 - x}; \text{ whence } x = 6. \text{ Ans.}$$

2. The first two terms of a harmonical progression are a and b ; prove that the third term is $\frac{ab}{2a - b}$.

3. Find the harmonical mean between 21 and 28.

Let x = the harmonical mean.

then $\frac{21}{28} = \frac{21 - x}{x - 28}$; whence, $x = 24$, Ans.

4. Prove that the harmonical mean between a and c is $\frac{2ac}{a + c}$.

412. All problems relating to quantities in harmonical progression, which are susceptible of solution, may be solved by inverting the terms and applying the rules of arithmetical progression. It should be remembered, however, that there is no general rule for finding the sum of the terms.

5. Find the 26th term of the series

$$\frac{4}{3}, \frac{3}{2}, \frac{1^2}{7}, \text{etc.} \dots \dots \dots (1).$$

Inverting the terms and we have

$$\frac{3}{4}, \frac{2}{3}, \frac{7}{1}, \text{etc.} \dots \dots \dots (2)$$

which is an arithmetical progression, in which the common difference is $\frac{2}{3} - \frac{3}{4} = -\frac{1}{12}$.

Now (397), the 26th term of (2) is

$$\frac{3}{4} - (26 - 1) \frac{1}{12} = -\frac{4}{3}.$$

Hence, the 26th term of (1) is $-\frac{3}{4}$, Ans.

6. Insert 3 harmonical means between 2 and 10.

Inverting, we have to insert 3 arithmetical means between $\frac{1}{2}$ and $\frac{1}{10}$.

Art. 399, $d = (\frac{1}{10} - \frac{1}{2}) \div 4 = -\frac{1}{10}$.

Hence, the arithmetical means are $\frac{2}{5}, \frac{3}{10}, \frac{1}{5}$,
and the required harmonical means $2\frac{1}{2}, 3\frac{1}{3}, 5$.

7. Insert 2 harmonical means between 3 and 12.

8. Find the 23d term of $\frac{5}{3}, \frac{3}{2}$, etc.

-
9. Insert 6 harmonical means between 3 and -1 .
10. The first term of a harmonic series is $\frac{1}{2}$, and the 6th $\frac{1}{12}$; find the intermediate terms.
11. Find the next two terms of the series $3, \frac{6}{5}, \frac{3}{4}$.
12. The arithmetical mean of two numbers is 5, and the harmonical mean is $3\frac{1}{6}$, find the numbers.

If a, b and c are in :

13. Arithmetical progression, prove that $b^2 > ac$.
14. Geometrical progression, prove that $b^2 = ac$.
15. Harmonical progression, prove that $b^2 < ac$.
16. If a, b, c are in arithmetical progression, and b, c, d are in harmonical progression, prove that a, b, c, d are in geometrical progression.
17. If any three quantities are in harmonical progression, prove that the three quotients obtained by dividing each of the quantities by the sum of the other two, are also in harmonical progression.

ANSWERS.

Page 10. 1, + 5, - 8; 2, + 65, - 50; 3, + 75, - 22; 4, + \$11, - \$13; 5, - 3° , + 5° ; 6, + 17° , - 15° ; 7, A boy went east 4 mi., then west 3 mi., then west 2 mi.; 8, A boy went north 4 mi., then south 3 mi., then south 2 mi.; 9, A gain of \$5, then a gain of \$4, then a loss of \$7, then a loss of \$1.

Page 12. 1, - 6; 2, - 1; 3, + 14; 4, 0; 5, + 2; 6, - 3; 7, - 10; 8, + 5.

Page 13. 9, + 12; 10, - 20; 11, + 9; 12, + 12; 13, + 2; 14, - 3; 16, - 5; 17, + 10; 18, + 3; 19, + 13.

Page 14. 30, 1 yr.; 31, 2 yr. younger; 32, 1 mi. east; 33, 4° lower; 34, 12° south; 36, 1 mi. east; 37, \$9.

Page 16. 1, + 4; 2, + 14; 3, - 17; 4, + 2; 5, - 7; 6, - 4; 7, + 22; 8, - 10; 9, + 6; 10, + 20; 11, - 23; 12, - 14; 13, + 17; 14, + 4; 15, + 2; 16, + 21; 17, - 5; 18, - 20; 19, + 185; 20, - 240.

Page 17. 23, 25 mi. east; 25, \$111; 26, \$10; 27, 373; 29, - 26 ft. or 26 ft. west; 30, - 71 ft. or 71 ft. west; 31, + 35 ft. or 35 ft. east.

Page 18. 32, + 16; 33, + 12; 34, - 27; 35, + 27; 36, - 4; 37, + 5; 38, - 10; 40, + 13; 41, + 7.

Page 19. 1, - 24; 2, - 21; 3, + 56; 4, + 54; 5, + 442; 6, - 714.

Page 20. 9, — 12; 10, + 360; 11, + 720; 12, + 240; 13, — 27; 14, — 12; 15, — 16; 16, — 117; 17, — 120; 18, — 720; 19, — 945; 20, + 240; 21, + 120; 22, + 168.

Page 21. 1, + 6; 2, + 5; 3, — 9; 4, — 7; 5, — 12; 6, — 25; 7, — 28.

Page 22. 8, — 5; 9, — 12; 10, + 5; 11, + 4; 12, — 30; 13, — 81; 14, + 7; 15, — 56; 16, — $20\frac{1}{2}$; 17, — $3\frac{1}{4}$; 18, + $6\frac{2}{3}$; 19, — $7\frac{1}{3}$; 20, + $1\frac{1}{2}$; 21, + 4; 22, — 2.

Page 23. 3, — 216; 4, + 2401; 5, + 15,625; 6, + 256; 7, — 512; 8, + 243; 9, + 90; 10, + 280; 11, — 800; 12, + 72; 13, — 3; 14, + 2,800,000.

Page 24. 2, ± 6 ; 3, ± 3 ; 4, + 5; 5, — 2; 6, — 4; 7, ± 4 ; 8, ± 7 ; 9, — 5; 10 ± 6 .

Page 25. 1, + 27; 2, — 16; 3, + 6; 4, + 51; 5, 0; 6, + 9; 7, — 78; 8, + 16; 9, + 16; 10, + 250; 11, + 9; 12, ± 4 ; 13, + 20; 14, + 2; 15, 4; 16, $2\frac{1}{4}$; 17, 1; 18, 2.

Page 27. 3, $7 \times a \times b \times b \times b \times n$; 4, $5 \times a \times a \times c \times c \times c \times c \times x \times x$; 5, $3 \times n \times n \times n \times n \times g \times y \times y \times y$; 6, $5 \times (a + b) \times x \times x \times y$.

Page 30. 1, 53; 2, 28; 3, 13; 4, 52; 5, 118; 6, 10; 7, 14; 8, 805; 9, 62; 10, — 5; 11, $4\frac{1}{3}$; 12, $\frac{1}{3}\frac{1}{4}$.

Page 32. 2, $(a + b) + (a - b) = 2a$; 3, $\frac{1}{2}(a + b) - \frac{1}{2}(a - b) = b$; 4, $(a + b) \times (a - b) = a^2 - b^2$; 5, $(a + b)^2 = a^2 + b^2 + 2ab$; 6, $(a - b)^2 = a^2 + b^2 - 2ab$.

Page 33. 2, $8a - 2b + 4c$; 3, $16a^2 + 7bc - 12$; 4, $5a^3 - 14a^2 + 16a + 1$; 5, $5a^4 - 20a + 7$; 6, $2xy^2 + 2x^2y + 7x^2y^2$.

Page 34. 7, $10a^3b + a^2b^2 - 3ab^3$; 8, $2a^2b + b^3$; 9, $9x^2y + 5xy^2 + 11x^2 - 10x + 4y - 3x^2y^2 + 3$; 11, $10\sqrt{abc}$; 12, $16(a - b)$; 13, $10\sqrt{ac} + 3\sqrt{m} + 2x$; 14, $2\sqrt{a - b} - 14\sqrt{a + c}$; 15, $-6a^2 + 5(a - n)^2$; 16, $2xy + 33x^2z - 19xz^2$; 17, $30a - 3b - 26c + 4f + g$; 18, $6(a - c)^2 - 2a\sqrt{6} - 7c^3x$; 19, $2x^2 - ac + 2(a + b)$.

Page 35. 2, $-2x^2 - 9$; 3, $7b + 2a - 9c$; 4, $12an - 12ax + 14$; 5, $2ab$; 6, $12x^2y$; 7, $-12ab^2$; 8, $-5xz$; 9, $m - n$; 10, $m + n$; 11, $-m - n$; 12, $-m + n$; 13, $11a^2x$; 14, $-14xy$; 15, $3\sqrt{3}$; 16, $12\sqrt{a}$; 17, $2b$; 18, $10a - 10x + 7y$.

Page 36. 19, $10c$; 20, $10\sqrt{m}$; 21, $3x^2$; 22, $b - a$; 23, $-a + c$; 24, $8a$; 25, $-2b$; 26, $2y$; 27, $-6x^2 + 13x - 2$; 28, $2\sqrt{a} - 2\sqrt{b} - 5\sqrt{5}$; 29, $-2a + 8b - c$; 30, $-a^2 - 7bx + 3x^2$; 31, $2b$; 32, $2a - 4c$; 33, $5x^2 - y - 2b$; 34, $x + x^3$; 35, $9a - 5b$; 36, $3x$; 37, $5a$; 38, $3x + 2$; 39, $-7\sqrt{2}$.

Page 37. 40, $(a - b) + (2c - 3d + 4e - 5f + x - 5mn)$; 41, $(a - b + 2c) - (3d - 4c + 5f - x + 5mn)$; 42, $(a - b + 2c - 3d) + (4e - 5f + x - 5mn)$; 43, $(a - b + 2c - 3d + 4c) - (5f - x + 5mn)$; 44, $(a + 2c + 4e + x) - (b + 3d + 5f + 5mn)$; 45, $(a + b) - (a - b)$ or $2b$; 46, $(a - b) - (a + b)$ or $-2b$; 47, $2(2x^2 + 7x - 4)$; 48, $-2(b^2 + 3ab - 5b^2)$.

Page 38. 2, $18a^6$; 3, $35a^5b$; 4, $-24a^7bx$; 5, $+45ab^5x^2$; 6, $56a^3c^2n^2x^5$; 7, $-36x^5y^3z^6$; 8, a^5 ; 9, a^{m+n} ; 10, a^{5x} ; 11, a^{2c} ; 12, $-21a^{n+2}$; 13, $15a^{m+2}c^{n+5}$; 14, $72a^8b^5c^6n^2x^2$; 15, $-42a^{3n}b^{4m}c^{x+1}$; 16, $60(a + b)^5x^7$; 17, $-77(a + b)^4(a - b)^9$; 18, $32(a + 2)^7(a + 1)^5(a + b)(a - b)^2$; 19, $-35(a - b + c)^5$; 20, $54a^5b^8c^4x^2y$.

Page 39. 23, $20a^5 - 12a^4 + 16a^3 - 8a^2$; 24, $-35x^7 - 25x^6 + 15x^5 + 40x^4$; 25, $4a^5x^2 - 12a^4x^3$

+ $12a^3x^2 - 4a^2x$; 26, $-9a^5b^2 + 18a^4b^3 - 21a^3b^4 + 6a^2b^5$; 27, $a^2b^3c^6 - a^2b^4c^5 + a^2b^5c^4 - a^2b^6c^3$; 28, $14a^7x^3y - 12a^6x^4y + 10a^5x^5y - 8a^4x^6y$; 29, $15(a+b)^5 + 18(a+b)^4$; 30, $-30a^6(a-b)^6 + 35a^7(a-b)^7$; 31, $28(x+y)^7(x-y)^7 - 36(x+y)^5(x-y)^{10}$; 32, $20a^{n+2} - 24a^{n+2} + 28a^7$; 33, $15a^{n+3} - 36a^{n+7} + 9a^{2n}$; 34, $-54a^{n+3}b^{2n+2} + 90a^{n+5}b^{3n} - 63a^{2n}b^{2n+5} + 27a^{3n}b^{5n}$.

Page 40. 37, 55; 38, 35; 39, -27 ; 40, -36 ; 41, 63; 43, $a^2 + 2ab + b^2$; 44, $a^2 - 2ab + b^2$; 45, $a^2 - b^2$; 46, $25a^2 + 30ab + 9b^2$; 47, $49c^2 - 84cx + 36x^2$; 48, $64a^4 - 49b^2$; 49, $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$; 50, $x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$.

Page 41. 51, $a^4 - 36a^2c^2 + 81c^4$; 52, $x^3 - y^3$; 53, $a^3 + 27y^3$; 54, $a^{10} = 1$; 55, $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$; 56, $a^4 + a^3b + a^3c + a^2b^2 + a^2c^2 + a^2bc + ab^2c + ac^2b + b^2c^2$; 57, $x^3 - y^3 - z^3 - 3x^2y - 3x^2z + 3xy^2 + 3xz^2 - 3x^2y - 3y^2z + 6xyz$; 58, $a^3 + 3a^2b + 3ab^2 + b^3 - 1$; 59, $8x^6 - 36x^4y + 54x^2y^2 - 27y^3 + 1$; 60, $x^4 - x^3 - 11x^2 + 9x + 18$; 61, $x^4 - 20x^2 + 64$; 62, $x^4 - x^3 - 18x^2 - 6x + 36$; 63, $x^4 - 256$; 64, $x^8 + x^4 + 1$; 65, $x^3 + ax^2 + bx^2 + cx^2 + abx + acx + bcx + abc$; 66, $x^5 + 36x^3 - 10x^4 - 86x^2 + 131x - 168$; 67, $a^6 - b^6 + 3a^2b^4 - 3a^4b^2$; 68, $a^2 + 2ab + b^2$; 69, $a^3 + 3a^2b + 3ab^2 + b^3$; 70, $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$; 71, $x^4 + 4x^3 + 4x^2$; 72, $x^6 - 6x^5 + 9x^4$; 73, $16x^4 + 96x^3 + 216x^2 + 216x + 81$; 74, $125a^3 + 450a^2c + 540ac^2 + 216c^3$; 75, $a^8 - 8a^6 + 24a^4 - 32a^2 + 16$; 76, $4a^6 - 20a^5b + 25a^4b^2$.

Page 42. 3, $x^2 + 10x + 21$; 4, $x^2 + 14x + 48$; 14, $x^2 - 18x + 80$; 15, $x^2 - 32x + 240$.

Page 43. 25, $x^2 + 7x - 18$; 32, $x^6 + 2bx^3 - 120b^2$; 34, $x^2 - 7x - 18$; 35, $x^3 - 15x - 100$.

Page 44. 45, 64; 47, 16; 49, $x^2 - 12x + 36$; 50, $x^3 - 36$; 51, $4x^3 + 4x + 1$; 53, $a^2 + 18a + 81$; 55, $a^2 - 81$; 56, $a^4 + 14a^2 + 49$; 58, $a^4 - 121$; 60, $x^2 - 6ax + 9a^2$.

Page 45. 61, $y^4 + 12ay + 36a^2$; 63, $9x^4 + 12x^3 + 4x^2$; 65, $9a^2x^2 + 36a^3x + 36a^4$; 67, $81 - x^2$; 69, $x^2 - 18bx^3 + 81b^3$; 71, $25x^4 - 60bx^3 + 36b^2x^2$; 73, $36n^6x^2 - 120n^5x^3 + 100n^4x^4$; 75, $25x^4 - 9y^4$; 77, $a^4b^6 - a^3b^8$; 78, $1 - 144a^8$; 79, $25a^6x^4y^2 - 1$; 80, $a^8 - 81$.

Page 46. 3, 16; 4, 100; 5, 16; 6, $x^4 - 4x^3 + 4x^2$; 7, $x^4 + 4x^3 - 2x^2 - 12x + 9$; 8, $x^6 - 6x^5 + 9x^4 - 10x^3 + 30x^2 + 25$; 9, $4a^4 - 12a^3b + 25a^2b^2 - 24ab^3 + 16b^4$; 10, $a^2 - 2ab + 2ac + c^2 - 2bc + b^2$; 11, $a^2 + b^3 + c^2 - 2ab - 2ac + 2bc$; 12, $a^4 + 2a^3 + a^2 - 8a^2x - 8ax + 16x^2$; 13, $4x^6 - 12x^5 + 25x^4 - 28x^3 + 22x^2 - 8x + 1$; 14, $4 - 4a + 9a^2 - 8a^3 + 6a^4 - 4a^5 + a^6$; 15, $a^4 - 4a^3b + 16a^2b^2 - 24ab^3 + 36b^4$; 16, $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1$.

Page 47. 1, $3a^3b$; 2, $-3x$; 3, $5a^3$; 4, $-3a^4c$; 5, $5a^2n^3$; 6, $2b^3d^3$; 7, $-3n^3$; 8, $-6abc$; 9, $2a^3bc^3$; 10, ac^5x ; 12, $b^0 = 1$; 13, $c^0 = 1$; 14, $(a + b)^0 = 1$; 15, $x^0 = 1$.

Page 48. 17, 4; 18, -1 ; 19, $-2a + 4a^2b - 5a^3c$; 20, $2ay + 6a^2xy^2 - 1$; 21, $-5 + 7ab^3y - 3a^2b^3nx$; 22, $5acm + 7 - 10a^2cb$; 23, $2a^2 - 3b^4 + 4c^6$; 24, $4(a + b)^3 - 2(a + b) + m$; 25, $-4a(a - x) - 3c(a - x)^4 + 5$; 26, $-6(a - x)^2(b + y) + c(b + y)^3$.

Page 50. 30, 5; 31, — 2; 32, 15; 33, — 2; 34, 39; 35, $x + 6$; 36, $x - 7$; 37, $x - 5$; 38, $x + 4$; 39, $x - 7$; 40, $x - 9a$; 41, $x - 2a^2b$; 42, $x^2 + 3x + 9$; 43, $x^2 - 4$; 44, $ay^2 + 3bx^3$; 45, $a^2 + ab + b^2$; 46, $a^3 + a^2b + ab^2 + b^3$; 47, $x^4 + 5x^2 + 13x + 14$; 48, $a^2 - ax + x^2$; 49, $x^3 - 2x^2y + 2y^3$.

Page 51. 50, $2a^2 + 5ax + x^2$; 51, $x^3 - x^2y + xy^2 - y^3$; 52, $24x^2 - 2ax - 35a^2$; 53, $20a^2 - 80ac + 15c^2$; 54, $a - b$; 55, $3a^4 + 3a^2y^2 + 3y^4$; 56, $a^3 + 3a^2b + 3ab^2 + b^3$; 57, $x^2 - xy + y^2$; 58, $4a^4b^2 + 2a^3b^3 + a^2b^4$; 59, $8a^9b^6c^3 - 4a^7b^6c^5 + 2a^5b^6c^7 - a^3b^6c^9$; 60, $x^4 + 2x^3 + 3x^2 + 2x + 1$; 61, $a^2 - 2ab + b^2$; 62, $x^2 + y^2 + z^2 - xy - xz - yz$; 63, $z^2 - 16z + 63$; 64, $5a^4 - 2a^3b + 4a^2b^2$; 65, $x^2 + 3x + 6 + \frac{5(2x - 1)}{x^2 - 2x + 1}$; 66, $x + a + 4$; 67, $x - c - 5$; 68, $x + n + 4$; 69, $x + n - 5a$; 70, $(x + a)^2 + 4(x + a) + 16$; 71, $x + a - c + 8$; 72, $(x + c)^2 - (x + n)^2$.

Page 53. 5, 0; 6, 12; 7, — 20; 8, 0; 9, 0; 10, — 12; 11, 64; 12, 0.

Page 54. 13, 3; 14, — 60; 15, yes; 16, yes; 17, no; 18, yes.

Page 55. 1, $a^2 + a + 1$; 3, $a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6$; 6, $4 + 6x + 9x^2$; 9, $5a^2x + 3$.

Page 56. 13, $a^2 - ax + x^2$; 15, $4 - 2a + a^2$; 18, $a^6 - 5a^3 + 25$; 23, $256a^4 - 192a^3b + 144a^2b^2 - 108ab^3 + 81b^4$.

Page 57. 25, $a^3 + a^2 + a + 1$; 26, $a^3 - a^2 + a - 1$; 29, $8x^3 - 4x^2 + 2x - 1$; 31, $a^3x^6 + a^2x^4 + ax^2 + 1$;

35, $27x^3 - 18x^2y + 12xy^2 - 8y^3$; 39, $x - 2$; 40, $(a + 2)^3 + (a + 2)^2 2x + (a + 2)(4x^2) + 8x^3$; 41, $x^2 + 5x + 1$; 42, $a^3 - a^2(3a^2 - 2a + 1) + a(3a^2 - 2a + 1)^2 - (3a^2 - 2a + 1)^3$.

Page 58. 2, $x(a + x)$; 3, $xy(x - y)$, 8, $m(a + b)$; 9, $b(a^2 + c^2 - bd)$; 10, $ac(6a - 3b + 7ac)$; 15, $(a + x) \times (a + b)$.

Page 59. 19, $(x + a)(x + c)$; 21, $(a + m)(b - n)$; 22, $(a - m) \times (b - n)$; 25, $(a^2 + n)(b^2 - m)$; 28, $(x + 3)(x + 4)$; 31, $(x + a) \times (x - b)$; 34, $(y + 2)(y - 5)$.

Page 60. 2, $(x + 9)(x - 9)$; 5, $(9a^2b + 12)(9a^2b - 12)$; 8, $(a^2 + 20b)(a^2 - 20b)$; 15, $(1 + 15a^3b^2c)(1 - 15a^3b^2c)$; 16, $(x^2 + 1) \times (x + 1)(x - 1)$; 17, $(1 + 4a^4)(1 + 2a^2)(1 - 2a^2)$; 19, $(a^8 + b^8)(a^4 + b^4)(a^2 + b^2)(a + b) \times (a - b)$; 21, $(x + a + n)(x + a - n)$; 22, $(x - b + m)(x - b - m)$; 23, $(x + 7)(x - 3)$; 24, $y(12 - y)$; 26, $(2x)(2a)$; 27, $(x^2 + 4x + 1)(x^2 - 1)$; 30, $(x - a + 3b)(x - a - 3b)$.

Page 61. 32, $(a - b)(a^2 + ab + b^2)$; 33, $(x - 2)(x^2 + 2x + 4)$; 34, $(x - 3)(x^3 + 3x^2 + 9x + 27)$; 35, $(y - 4)(y^2 + 4y + 16)$; 39, $(2x - 3y)(4x^2 + 6xy + 9y^2)$; 41, $[(x + 1) - 2][(x + 1)^3 + 2(x + 1)^2 + 4(x + 1) + 8]$; 45, $(x + 2) \times (x^2 - 2x + 4)$; 46, $(x + 3)(x^4 - 3x^3 + 9x^2 - 27x + 81)$; 47, $(x + 4)(x^2 - 4x + 16)$; 50, $(4b + 5)(16b^2 - 20b + 25)$; 52, $[(x + 1) + 3][(x + 1)^2 - 3(x + 1) + 9]$.

Page 62. 3, $(x + 3)(x + 6)$; 4, $(x + 2)(x + 7)$; 8, $(x + 4a)(x + 8a)$; 11, $(y^2 + 8a^2)(y^2 + 8a^2)$.

Page 63. 20, $(x - 1)(x - 10)$; 21, $(x - 2)(x - 9)$; 28, $(x - 7a)(x - 9a)$; 32, $(y^3 - 5b)(y^3 - 7b)$; 37,

$(x-2)(x+7)$; 38, $(x-3)(x+8)$; 45, $(x-8a)(x+15a)$; 47, $(x^2-15c)(x^2+20c)$.

Page 64. 54, $(x+2)(x-7)$; 55, $(x+3)(x-8)$; 63, $(x+8n)(x-13n)$; 69, $(y^3z^4+10)(y^3z^4-11)$.

Page 65. 3, $(x+7)^2$; 4, $(x-8)^2$; 13, $(1-3ab^2)^2$; 20, $(18a^2c-ac^3)^2$.

Page 66. 3, $(3x+1)(2x+3)$; 4, $(3x-8)(4x+5)$; 5, $(2x-1)(5x-3)$; 6, $(4x-5)(5x-4)$; 9, $(6x-5)(3x+2)$; 10, $(6x-5)(4x+1)$.

Page 67. 2, $(a^2+a+1)(a^2-a+1)$; 3, $(3a^2+3ac+5c^2)(3a^2-3ac+5c^2)$; 5, $(7a^2+13ab+11b^2)(7a^2-13ab+11b^2)$; 7, $(4x^2+3xy-7y^2) \times (4x^2-3xy-7y^2)$; 9, $(5x^2+xy-4y^2)(5x^2-xy-4y^2)$; 13, $(12+x-9x^2) \times (12-x-9x^2)$.

Art. 158. 1, $(x-1)(x+1)$; 2, $(x-7)(x-8)$; 3, **Art. 151**; 5, **Art. 150**; 6, $(x-11)(x+4)$; 10, **Art. 154**; 12, $(3x+4)(2x-1)$; 15, $x(x+a)(x^2-ax+a^2)$; 16, $(3a-2)(2a-5)$; 17, $a^2(a-b)(a^2+ab+b^2)$; 18, **Art. 156**; 19, $5(1+2x)(1-2x+4x^2)$; 20, $(2x^2+x+1)(2x^2-x+1)$; 21, $7(2x-3)(4x^2+6x+9)$; 22, **Art. 156**; 23, $x(8x^3+1)(4x^2+2x+1)(2x-1)$; 26, $(5x-1)(x^2+1)$; 27, $3(x^2-5)(x^2-8)$; 28, **Art. 154**; 29, $x(x-3)(x+5)$; 30, **Art. 154**; 33, $5(x^4+7)(x^4-12)$; 36, $(p-q)(x+y)(x-y)$.

Page 68. 37, $7(x^2+5)(x^2-12)$; 38, $(x^2+1)(x-1)$; 40, $x(x+1)(x-1)(x-3)$; 41, $x(8x-10)(3x+8)$; 42, **Art. 147**; 43, **Art. 154**; 44, **Art. 148**; 46, **Art. 147**; 47, $a^3(a^2+a+1)(a^2-a+1)$; 48, $(a+b+c)(a+b-c)(c-a+b)(c+a-b)$; 49, $y^2(3x+2y)(x-y)$; 50, $(x+y-z)(x-y+z)(x+y+z)(x-y-z)$.

Page 69. 3, $2a^2b$; 5, $4b^2x$; 8, $13bc$; 10, $a(a+b)^2$; 13, $x+y$; 14, $2x-1$; 17, $x-c$; 18, $2(x-y)^3$; 19, $x+4$; 20, $x-5$; 21, $x+9$; 23, $x+9$; 25, $x-7$.

Page 70. 28, $x+y$; 29, $x-2$; 30, $a-3b$.

Page 73. 35, $x+7$; 36, $x+11$; 37, $3(x-1)$; 38, $2x+5$; 39, $4x-1$; 40, $a+b$; 41, $3x^2+12x$; 42, $a+x+y$; 43, $y+z$; 44, $a-3b$; 45, $x-2$; 46, $x-3$; 47, $2x-1$; 48, $2x^2+7x+3$; 49, $(x+1)^3$.

Page 74. 3, $24a^3b^2c$; 4, $a^4b^3c^2x$; 5, $60c^2x^3y^3$; 8, $a^2(a+b)^3$; 9, $48a^4b^3c^5$; 11, $(x+1)(x+2)(x-2)$; 12, $(a+b)(a-b)^2$; 13, $(x^2-5x+6)(x+3)$; 14, $(3+2x)^2(3-2x)$.

Page 75. 17, $(x-3)(x+3)(x+5)(x-7)$; 18, $(x-5)^2(x+6)^2$; 19, $30(a+b)^3(a-b)^3$; 20, $12a(a+b+c)^2(a+b)^3(a-b)$; 22, $(x^2-23x+112)(x+8)$; 23, $(x^2+28x+187)(x-13)$; 24, $(a-b)^2(a^2+b^2)(a+b)$; 25, $(x^3+6x^2+11x+6)(x+4)$; 26, $(x^4-16)(x^4-81)$; 27, $(x^2-y^2)(x^2-4y^2)$.

Page 79. 3, 20: 4, x^3+x-6 ; 5, 33; 6, $15(a^2-b^2)$; 7, $5(a+1)$; 8, $7c^2(a+1)$; 9, $2bx$; 10, $(x+2)^2$; 11, $3a^2bcx$; 12, y^2-25 ; 13, x^3-a^3 ; 14, $a^3-2a^2b+2ab^2-b^3$; 15, $18a^5b^2cn$; 16, $x^6+2x^4y^3+2x^2y^6+y^9$,

Page 80. 18, 62; 19, $12x^2+18x+10$; 20, 78; 21, $6x^3+4x^2-7x-3$; 22, 192; 23, $5x^3-18x^2+32$; 25, 45; 26, 43; 27, $3x-17$; 28, $x^3-47x^2+224x-48$.

Page 81. 30, 15; 31, am ; 32, 91; 33, $15x^2$; 34, x^3-x^2-6x ; Art. 185. 1, $\frac{bc}{2a}$; 2, $\frac{4x^3y}{5a^2z}$; 3, $\frac{(a+x)^2}{a+b}$;

$$4, \frac{5b^2}{9c^2x}; \quad 5, \frac{a^2(1+x)(1-x)^2}{3}; \quad 6, \frac{3n^5}{5a^2m};$$

$$7, \frac{3c^2(a+x)^4}{5a^3(a-x)^3}.$$

$$\text{Page 82. } 9, \frac{x+8}{x-4}; \quad 10, \frac{x-8}{x-10}; \quad 11, \frac{x+3}{x-9};$$

$$12, \frac{x-7}{4x}; \quad 13, \frac{x+9}{x+6}; \quad 14, \frac{a+1}{a-7}; \quad 15, \frac{x+1}{x^2+2x-3};$$

$$16, \frac{x^2-4x+3}{x-4}; \quad 17, \frac{a^2-2a+2}{a^2-2}; \quad 18, \frac{2x-5}{3x-11};$$

$$19, \frac{2x+5y}{2x+3y}; \quad 20, \frac{a-3}{a-2}. \quad \text{Art. 186. } 1, x+2; \quad 2, 2x$$

$$+ 6y + \frac{65y^2}{3x-10y}; \quad 3, 5x-17 + \frac{69}{x+4}; \quad 4, a^2 + 2ab$$

$$+ 5b^2 + \frac{8ab^3 - 4b^4}{a^2 - 2ab + b^2}; \quad 5, 7a - 47 + \frac{289}{a+6};$$

$$6, 1 - \frac{12}{x+4}.$$

$$\text{Page 83. } 7, a^2 - ab + b^2; \quad 8, x+4 - \frac{20}{x+5}; \quad 9, a^3 - a^2b + ab^2 - b^3 + \frac{2b^4}{a+b}; \quad 10, 6x-13 + \frac{10}{x+1}.$$

$$\text{Art. 187. } 4, \frac{21}{36}; \quad 5, \frac{a^2+2a+1}{a^2-1}; \quad 6, \frac{a^2bnx^2}{n^2x^3};$$

$$7, \frac{2x^2-5x-3}{x^2-9}; \quad 8, \frac{10ac}{12bx}; \quad 9, \frac{x^2-8x+15}{x^2+x-12};$$

$$10, \frac{x^2-22x-15}{x^2+x-30}.$$

$$\text{Page 84. } 2, \frac{2x^2+4x+1}{x}; \quad 3, \frac{2x^2+8}{x+2};$$

$$4, \frac{2x^2 - 2x - 1}{x}; 5, \frac{12x}{x+3}; 6, \frac{x}{ab}; 7, \frac{x^2 + x - 2}{x+1};$$

$$8, \frac{2x(x^2 - x - 1)}{x-2}; 9, \frac{a}{a-1}.$$

Page 85. 3, $\frac{20}{30}, \frac{18a}{30}, \frac{21x}{30}; 4, \frac{6x-15}{24}, \frac{6x+14}{24};$

$$5, \frac{35abx}{15ab}, \frac{15a^2}{15ab}, \frac{3bc}{15ab}; 6, \frac{2ab+4b^2}{6ab}, \frac{2a^2-3ab}{6ab};$$

$$7, \frac{3m^2y}{3my}, \frac{3ny}{3my}, \frac{mx}{3my}; 8, \frac{(a+b)^2}{a^2-b^2}, \frac{(a-b)^2}{a^2-b^2};$$

$$9, \frac{60ax}{24a^2x}, \frac{56x}{24a^2x}, \frac{27a^2}{24a^2x}; 10, \frac{4x^2+12xy+9y^2}{4x^2-9y^2},$$

$$\frac{4x^2-12xy+9y^2}{4x^2-9y^2}; 11, \frac{9(1+x)^2}{8(1+x)^3}, \frac{6x(1+x)}{8(1+x)^3},$$

$$\frac{4x^2}{8(1+x)^3}; 12, \frac{x^3}{x^3-9x}, \frac{x^2-x-6}{x^3-9x}, \frac{x^2-9}{x^3-9x};$$

$$13, \frac{x^2+6x+9}{(x+1)(x+2)(x+3)}, \frac{x^2+4x+4}{x^3+6x^2+11x+6},$$

$$\frac{x^2+2x+1}{x^3+6x^2+11x+6}.$$

Page 87. 5, $\frac{17x}{12}; 6, \frac{2x+8}{x-x^3}; 7, \frac{x}{24}; 8, \frac{3x-8}{x^2-9};$

$$9, \frac{3x+1}{10}; 10, -\frac{x-9}{x^2-16}; 11, \frac{41-x}{28}; 12, \frac{2a^2}{a^2-b^2};$$

$$13, \frac{19x}{24}; 14, -\frac{2ab}{a^2-b^2}; 15, \frac{3bx-2ax+3a}{4ab};$$

$$16, \frac{2(a^2+b^2)}{a^2-b^2}; 17, \frac{11a}{3b}; 18, \frac{4ab}{a^2-b^2}.$$

Page 88. 19, $\frac{18x}{9x^2-1}$; 20, $\frac{4x^2+6x+3}{(x+1)^3}$;

21, $\frac{71x-51y}{50x}$; 22, $\frac{80x^3+64x^2+84x+45}{60x^2}$;

23, $\frac{2x^2+7x+1}{x^3+12x^2+47x+60}$; 24, $-\frac{11x+64}{(x+4)(x-5)(x+6)}$;

25, $\frac{x+6}{x^2+4x+3}$; 26, $\frac{x+7}{x^2+6x+8}$;

27, $\frac{2}{(x-4)(x-5)(x-6)}$; 28, $-\frac{2}{4x^3-x}$;

29, $\frac{2x^2+8x-1}{(1-x)(x+2)^2}$; 30, $\frac{1}{x+2}$; 31, 0.

Page 89. 32, $\frac{2x-3}{(x^2-1)(2x+3)}$; 33, $\frac{ax-b^2}{x^2-b^2}$;

34, $\frac{81a-4b}{84}$; 35, $\frac{2ab^2}{a^4-b^4}$; 36, $\frac{4a}{a+x}$; 37, 0;

38, -1; 39, 0; 40, 0; 41, 0.

Page 90. 3, $\frac{11a}{28b}$; 4, 5; 5, $\frac{3(x+a)}{10(x-a)}$;

6, $\frac{3}{4}(a^2+x^2)$; 7, $\frac{3}{8}\frac{5}{6}$; 8, $a+x$; 9, $\frac{1}{2}$; 10, $1\frac{1}{2}$;

11, $\frac{2a^2}{3x^2}$; 12, $\frac{9b}{2ax}$;

Page 91. 13, $\frac{2(a+x)^3}{3(x+y)^3}$; 14, $x^2-2x-15$;

16, $\frac{x^3-4x+3}{x^2-x-2}$; 17, $\frac{x^2+x-2}{x^2-3x-4}$; 18, 1;

19, 1; 20, $\frac{3x}{4y}$; 21, $\frac{1-y}{x}$; 22, $\frac{a^2+b^2}{a}$;

$$23, \frac{x^4}{a^4} + \frac{x^2}{a^2} + 1; \quad 24, x^2 + 1 + \frac{1}{x^2}; \quad 25, 1.$$

$$\text{Page 92. } 26, \frac{(10x^2 + 9x + 2)(2x - 1)}{(5x^2 + 23x + 12)(x - 4)};$$

$$27, \frac{x - 1}{x + 1}; \quad 2, \frac{9}{10}; \quad 3, \frac{3}{4}; \quad 4, \frac{10x - 5}{3x};$$

$$5, \frac{a - b}{a^2}; \quad 6, 2(x - 1); \quad 7, \frac{(a^2 + ax + x^2)(a + x)^2}{(a^2 - ax + x^2)(a - x)};$$

$$8, \frac{x + 3}{x - 4}; \quad 9, \frac{-(m + n)}{c^2 - 2c + 4}; \quad 10, -\frac{x(x + 1)}{2 + x}.$$

$$\text{Page 93. } 11, \frac{x^2 - 7x + 6}{x^2}; \quad 12, \frac{a}{b} - \frac{b}{a};$$

$$13, \frac{x^3 - y^3}{x^3 + y^3}; \quad 14, 1; \quad 15, \frac{x^4 - 18x^2 + 80}{x^4 + 18x^2 + 80};$$

$$16, x - \frac{1}{x}; \quad 17, \frac{x^2 - 12x + 35}{x^2 - 14x + 48}; \quad 18, x^2 + 1 + \frac{1}{x^2};$$

$$19, x + \frac{1}{x}; \quad 20, x + 1 + \frac{1}{x}; \quad 21, \frac{x^2 - 2a^2}{ax};$$

$$22, a^2 - b^2 + c^2 + 2ac; \quad 23, a^2 - b^2 + c^2 - 2ac$$

$$24, \frac{x^2 + 3ax - 2a^2}{x + 6a}.$$

$$\text{Page 94. } 3, \frac{81}{10}; \quad 4, \frac{x - 1}{x + 1}; \quad 5, \frac{31}{148};$$

$$6, \frac{(3x + 18)(2x - 3)}{(2x + 6)(2x + 3)}.$$

$$\text{Page 95. } 7, \frac{a^2 + x^2}{2ax}; \quad 8, \frac{x^2 + 1}{2x}; \quad 9, 1;$$

$$10, \frac{x^2 - 64}{x^2 - 144}; \quad 11, -\frac{a^4 + a^2b^2 + b^4}{ab(a - b)^2};$$

$$12, -\frac{bc(b - c)^2}{b^4 + c^4 + b^2c^2};$$

$$13, \frac{xy}{x^2 + y^2}; \quad 14, \frac{(a^2 + b^2)^2}{2a^2b^2}; \quad 15, \frac{x^2 + 1}{x^3 + 2x};$$

$$16, \frac{bc + 1}{abc + a + c}; \quad 17, \frac{adf + ae}{bdf + be + cf}; \quad 18, \frac{4}{3(x+1)};$$

Page 97. 11, 4; 12, 2; 13, 10; 14, 3; 15, 12; 16, 7.

Page 99. 2, 3; 3, 5; 4, $2\frac{1}{4}$; 5, $4\frac{2}{3}$; 6, $\frac{b}{a}$; 7, $\frac{c}{m}$;
 8, a ; 9, cm ; 10, $\frac{d}{a-b}$; 11, $\frac{m}{a+n}$; 12, $a-5$; 13, $n^2 + 3n + 9$; 15, 2; 16, 8; 17, 9; 18, $3\frac{1}{2}$; 19, -3 ;
 21, $\frac{m+r}{a+b-n}$; 22, $\frac{n+m-a}{m-n}$; 23, $a-b$; 24, $m+n$;
 25, $b^2 - bc + c^2$; 26, $c^2 - c + 1$; 27, $m+n$; 28, $a^2 + c^2$; 29, $a^2 - a + 1$.

Page 100. 31, 3; 32, -2 ; 33, $5\frac{1}{4}$; 34, $\frac{n-m}{a+b}$;
 35, $\frac{b-n}{c-m}$; 36, $c^2 + cn + n^2$; 37, $c^2 - cn + n^2$;
 38, 10; 39, 13; 40, 8; 41, 7; 42, -8 ; 43, 13; 44, $p+q$; 45, $a+b$.

Page 101. 47, 3; 48, $10\frac{4}{5}$; 49, 5; 50, 24; 51, 13; 52, 11; 53, 7.

Page 102. 54, 20; 55, 13; 56, 4; 57, 4; 58, 3; 59, $1\frac{1}{2}$; 60, 4; 61, $\frac{3}{7}$; 62, 7; 63, 1; 64, 3; 65, $\frac{7}{2}$; 66, -2 ; 67, 12; 68, 12; 69, 5.

Page 103. 70, $4\frac{1}{2}$; 71, $\frac{ab+c}{a}$; 72, $-\frac{1}{2}(a+c)$;
 73, $\frac{6a+3}{6a+2b}$; 74, $\frac{18a+b}{a+3}$; 75, $\frac{b}{2}(1-a^2)$;

76, $\frac{c + 2a + 2b}{2b}$; 77, $\frac{ac}{b}$; 78, $\frac{1}{2}(a + b)$; 79, $3\frac{3}{8}$;

80, $\frac{ab}{a + b}$.

Page 105. 2, \$3, \$9; 3, \$2.50, \$15; 4, \$1.28, \$5.12.

Page 106. 7, \$39, \$114; 8, \$4.15, \$7.85; 9, 76 ft., 97 ft.; 10, $\frac{1}{2}(s + d)$, $\frac{1}{2}(s - d)$; 12, 20, 127.

Page 107. 13, 15 yr., 18 yr.; 14, \$51, \$228; 15, $\frac{s - d}{t + 1}$, $\frac{ts + d}{t + 1}$; 17, 120; 18, 270; 19, 54; 20, 120.

Page 108. 21, $\frac{dmn}{n - m}$; 22, $\frac{mns}{m + n}$; 24, 40, 60; 25, 20 min. past 5; 26, 28, 18; 27, \$250, \$600; 28, 7, 8; 29, $\frac{ms}{m + n}$, $\frac{ns}{m + n}$.

Page 109. 30, $\frac{amn - ms}{n - m}$, $\frac{ns - amn}{n - m}$; 32, 9, 51; 33, $9\frac{4}{5}$, $10\frac{1}{5}$; 34, $\frac{s - r}{q + 1}$, $\frac{qs + r}{q + 1}$; 35, A £162, B £118, C £104; 36, 24, 36, 48; 37, 20, 30, 5, 125.

Page 110. 38, \$80; 39, \$90, \$180; 40, 20 yr.; 41, 18, 19; 42, 24, 25, 26; 43, 100.

Page 111. 45, 15 yr., 45 yr.; 46, 20 yr., 40 yr.; 47, 7 yr.; 48, 24 yr.; 49, $\frac{nr + m}{n - 1}$.

Page 112. 51, $3\frac{7}{11}$ da.; 52, $24\frac{6}{7}$ da.; 53, 15 da.; 54, 11 da., 22 da., 33 da.; 55, 6; 56, 30; 57, 12 hr.

Page 113. 58, A 32 da., B $19\frac{1}{2}$ da.; 59, 10 da.;
60, $\frac{abc}{ab + ac + bc}$.

Page 114. 62, 10; 63, 72; 64, $3\frac{3}{4}$ min., $6\frac{1}{4}$ min.;
65, 15 mi.; 66, 4 and 5 mi. per hour; 67, $\frac{abn}{b - a}$.

Page 115. 69, 420; 70, 600, 450; 71, 728.

Page 117. 73, 1° , $27\frac{3}{11}$ min. past 5, 2° , $43\frac{7}{11}$ min.
past 8, 3° , no time; 74, 1° , $21\frac{9}{11}$ min. past 1, 2° , $5\frac{5}{11}$
min. and $38\frac{2}{11}$ min. past 4, 3° , $21\frac{9}{11}$ min., and $54\frac{6}{11}$ min.
past 7; 75, 1° , $38\frac{2}{11}$ min. past 1, 2° , $54\frac{6}{11}$ min. past 4,
 3° , $10\frac{10}{11}$ min. past 8; 76, 1° , $43\frac{7}{11}$ min. past 7, 2° , $27\frac{3}{11}$
min. past 7; 77, 93.

Page 118. 78, $\frac{7}{10}$; 79, 70; 80, \$150; 81, \$20; 82,
6 miles; 83, 17; 84, 60; 85, 12, 7.

Page 119. 86, \$126; 87, 32 min. $18\frac{6}{13}$ sec. past 5;
88, A in 10, B in 12, C in 15, all in 4 days; 89, 15 mi.
per day; 90, $\frac{abc}{b + c}$; 91, hare $\frac{hnq}{mq - np}$, hound $\frac{hmq}{nq - np}$.
92, 1° , $7\frac{811}{1427}$ sec., 2° , $46\frac{338}{897}$ sec., 3° , $294\frac{3}{8}$ sec.

Page 123. 2, $x = 2$, $y = 3$; 3, $x = 16$, $y = 4$;
4, $x = 20$, $y = 10$; 5, $x = 6$, $y = 12$; 6, $x = 16$, $y = 7$;
7, $x = 7$, $y = 14$; 8, $x = 8$, $y = 4$; 9, $x = 5$, $y = 2$;
10, $x = 60$, $y = 30$.

Page 124. 12, $x = 1$; $y = 3$; 13, $x = 60$, $y = 36$;
14, $x = 5$, $y = 4$; 15, $x = 12$, $y = 20$; 16, $x = 5$,
 $y = 7$; 17, $x = -21\frac{7}{9}$, $y = 38\frac{1}{3}$; 18, $x = -6$, $y = 12$;
19, $x = 18$, $y = 6$.

Page 125. 21, $x = 2$, $y = 6$; 22, $x = 21$, $y = 15$;

23, $x = 3, y = 4$; 24, $x = 12, y = 9$; 25, $x = 4, y = 3$;
 26, $x = 3, y = 5$; 27, $x = 7, y = 11$; 28, $x = 13,$
 $y = 11$; 29, $x = 144, y = 216$; 30, $x = 4, y = 6$; 31,
 $x = 2, y = 4$; 32, $x = 2, y = -1$.

Page 126. 33, $x = .8, y = .4$; 34, $x = 5, y = 5$;

$$35, x = \frac{1}{2}(s + d), y = \frac{1}{2}(s - d); 36, x = \frac{3a + 2b}{17},$$

$$y = \frac{4a - 3b}{17}; 37, x = 7, y = 10; 39, x = \frac{qr - ns}{mq - np},$$

$$y = \frac{pr - ms}{mq - np}; 40, x = \frac{cn}{an + bm}, y = \frac{cm}{an + bm};$$

$$41, x = \frac{2}{a + b}, y = \frac{2}{a - b}; 42, x = \frac{2a}{c + d},$$

$$y = \frac{2b}{c - d}; 43, x = (a + b)^2, y = (a - b)^2.$$

Page 128. 2, $x = 3, y = 2, z = 5$; 3, $x = 10,$
 $y = 2, z = 3$; 4, $x = 2, y = 3, z = 4$; 5, $x = 3,$
 $y = 4, z = 6$; 6, $x = 2, y = 3, z = 4$; 7, $x = 5, y = 4,$
 $z = 3$; 8, $x = 1, y = 2, z = 3$; 9, $x = 8, y = 9, z = 12$; 10,
 $x = 4, y = 9, z = 16, u = 25$; 11, $x = 2, y = 3, z = 4,$
 $u = 5$; 12, $x = \frac{b^2 + c^2 - a^2}{2bc}, y = \frac{a^2 + c^2 - b^2}{2ac},$

$$z = \frac{a^2 + b^2 - c^2}{2ab}; 13, x = \frac{n(n - b)(n - c)}{a(a - b)(a - c)};$$

$$14, x = \frac{a}{2}, y = \frac{b}{2}, z = \frac{c}{2}; 15, x = b + c - a.$$

Page 129. 2, 5, 7; 3, 12, 15; 4, 85, 15; 5, \$250,
 \$320; 6, A \$42, B \$26.

Page 130. 8, 20, 32; 9, 42, 45; 10, 40, 60; 11,
 A \$180; B \$140; 12, 7, 10; 13, 1, 3, 5.

Page 131. 15, $-\frac{5}{28}$; 16, $\frac{4}{5}$; 17, $\frac{15}{20}$; 18, $\frac{2}{3}, \frac{1}{6}$; 19, $\frac{1}{4}, \frac{2}{3}$.

Page 132. 21, 36; 22, 48; 23, 245; 24, 891.

Page 133. 25, 39; 26, 853; 27, $\frac{1}{2}(11s - 9d)$; 28, \$90, \$120; 29, $\frac{(m+1)c}{mn-1}, \frac{(n+1)c}{mn-1}$; 30, 10 yr., 40 yr.; 31, 30 yr., 22 yr.

Page 134. 32, \$50, \$40; 33, A $6a + 5b + c$, B $\frac{3a + 5b + c}{2}$; 34, $\$3\frac{1}{2}, \$4\frac{1}{4}$; 35, tea 5s., sugar 4d.; 36, $18\frac{3}{4}$ lb., $31\frac{1}{4}$ lb.; 37, 40, 90.

Page 135. 39, 8, 12, 24 days; 40, 40 spel., 30 read.; 41, 16 h., 32 c.; 42, 18, 24, 36 days; 43, A in $\frac{2abc}{ac + bc - ab}$ min., B in $\frac{2abc}{ab + bc - ac}$ min., C in $\frac{2abc}{ab + ac - bc}$ min.

Page 136. 45, 5 mi. per hour; 46, Dist. 24 mi., rate 5 mi. per hour; 47, $\frac{bc}{c+d}, \frac{bd}{c+d}, \frac{a(c^2 - d^2)}{2bcd}$; 48, 10, 22, 26; 49, \$60, \$80.

Page 137. 50, 30, 20 rd., 600 sq. rd.; 51, A in 5 min., B in 6 min.; 52, 4 and 5 yd.; 53, 100 mi.; 54, Principals, \$3000, \$4000, \$4500; rates 4, 5, 6 per cent.

Page 138. 55, A \$21, B \$11; 56, 26, 14, 8 gal.

Page 140. 3, $x = 5, y = 2$; 4, $x = 5, 12$, etc., $y = 1, 5$, etc.; 5, $x = 10, y = 1$; 6, $x = 5, 10$, etc., $y = 2, 8$, etc.; 7, $x = 4, y = 5$; 8, $x = 10, 19$, etc.,

$y = 3, 16$, etc.; **9**, $x = 15$ or 2 , $y = 1$ or 2 ; **10**, $x = 8, 21$, etc., $y = 3, 8$, etc.; **11**, no values; **12**, $x = 2, 37$, etc., $y = 3, 46$, etc.

Page 142. **15**, 41, 59; **16**, it cannot; **17**, 23; **18**, two, viz: $\frac{5}{7}, \frac{6}{8}$ and $\frac{12}{7}, \frac{1}{5}$; **19**, $\frac{38}{7}, \frac{11}{2}$ and $\frac{16}{4}, \frac{32}{6}$; **20**, 40, 110, 180, etc.; **21**, $\frac{10}{30}, \frac{28}{2}$, etc.; **22**, 58; **23**, 6, 56, 8, or 12, 42, 16, or 18, 28, 24, or 24, 14, 32; **24**, 61.

Page 143. **25**, 34, 13, 1, or 37, 8, 3, or 40, 3, 5; **26**, 4, 11, 25, or 8, 2, 30; **27**, 4 ox., 6 h., 90 hens; **28**, 5 m., 3 w., 33 c.; **29**, A 1, B 15, C 29—each rec'd 65 ct.

Page 146. **1**, a^4 ; **2**, $243a^{10}c^5$; **3**, — $343a^9b^3z^6$; **4**, a^{12} ; **5**, $16x^{12}y^4$; **6**, $256b^8c^{4n}$; **7**, a^{4n} ; **8**, — $27x^6z^3$; **9**, — $243a^{5n}x^{15}y^5$; **10**, $a^{10}b^{15}$; **11**, $25a^8b^4$; **12**, $64a^{6m}b^{6n}c^6$; **13**, $\frac{a^8b^4}{81}$; **14**, $\frac{8x^5y^3}{125z^9}$; **15**, $\frac{625a^8b^4}{16c^{16}}$; **16**, $\frac{243x^5y^{15}}{1024}$; **17**, — $\frac{m^{21}}{n^{14}x^7}$; **18**, $\frac{a^{6n}b^{12}}{c^{6m}}$.

Page 147. **1**, $a^3 + 3a^2x + 3ax^2 + x^3$; **2**, $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$; **3**, $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$; **4**, $a^3 - 3a^2x + 3ax^2 - x^3$; **5**, $x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$; **6**, $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$; **7**, $a^3 + 3a^2 + 3a + 1$; **8**, $x^4 + 4x^3 + 6x^2 + 4x + 1$; **9**, $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$; **10**, $x^3 - 3x^2 + 3x - 1$; **11**, $x^4 - 4x^3 + 6x^2 - 4x + 1$; **12**, $y^5 - 5y^4 + 10y^3 - 10y^2 + 5y - 1$.

Page 148. **13**, $x^3 + 9x^2 + 27x + 27$; **14**, $x^6 - 6ax^4 + 12a^2x^2 - 8a^3$; **15**, $a^8 + 3a^5 + 6a^4 + 7a^3 + 6a^2 + 3a + 1$; **16**, $x^3 - 6x^2 + 12x - 8$; **17**, $a^8 - 12a^7 + 54a^6 - 108a^5 + 81a^4$; **18**, $x^4 - 2x^3 - x^2$

+ $2x + 1$; 19, $a^4 + 16a^3 + 96a^2 + 256a + 256$; 20, $a^6 + 12a^5c + 48a^4c^2 + 64c^3$; 21, $1 - 8a + 28a^2 - 56a^3 + 70a^4 - 56a^5 + 28a^6 - 8a^7 + a^8$; 22, $1 + 10a + 40a^2 + 80a^3 + 80a^4 + 32a^5$; 23, $16a^8 - 96a^6y + 216a^4y^2 - 216a^2y^3 + 81y^4$; 24, $125 - 150x - 15x^2 + 52x^3 + 3x^4 - 6x^5 - x^6$; 25, $32a^5 - 80a^4 + 80a^3 - 40a^2 + 10a - 1$; 26, $32a^{15} - 40a^{12}x + 20a^9x^2 - 5a^6x^3 + \frac{5}{8}a^3x^4 - \frac{x^5}{32}$; 27, $x^6 - 15x^5 + 81x^4 - 185x^3 + 162x^2 - 60x + 8$.

Page 149. 1, $\pm a^2$; 2, $-2a^2b$; 3, $\pm 12a^4b^3c$; 4, a^3 ; 5, $\pm 9x^3y$; 6, $6ax^3z^2$; 7, $4x^2$; 8, $3a^4b$; 9, $\pm 11a^5b^2x^3$; 10, $\pm 2a^2$; 11, $-5c^2y^5$; 12, $-a^7b^4c^3x$; 13, $\pm \frac{2x^2}{3a}$; 14, $\pm \frac{7a^2x^3}{8c^3}$; 15, $-\frac{2a^3c^3x^4}{3by^4}$; 16, 9; 17, 7; 18, $2a$; 19, $15a^2b$; 20, $10a$; 21, $-a^2x^3$.

Page 151. 3, $3ab^2 + 2$; 4, $8x^3 - 5xy$; 5, $10a^3b^2c + 1$; 6, $x^2 - x + 1$; 7, $x^2 - 2x - 2$; 8, $2x^2 + 3x - 1$; 9, $2x^2 - x + 1$; 10, $2x^2 - 3ax + 4a^2$; 11, $5x^2 - 3ax + 4a^2$; 12, $x^3 - 3ax^2 + 3a^2x - a^3$; 13, $x^2 - \frac{x}{2} + \frac{2}{x}$; 14, $\frac{a^2}{2} + \frac{a}{x} - \frac{x}{a}$.

Page 153. 3, 95; 4, 2625; 5, 7.82; 6, 125; 7, 1168; 8, 31.28; 9, 292; 10, 1404; 11, 280.9; 12, 234; 13, 1564; 14, .0733.

Page 154. 15, $\frac{12}{17}$; 16, $\frac{25}{31}$; 17, $\frac{34}{723}$; 18, $\frac{11}{2}$; 19, 37.97 +; 20, $291\frac{2}{3}$; 21, 1.4142 +; 22, .4714 +; 23, 11.157 +; 24, 1.7320; 25, .433 +; 26, 15.832 +; 27, 2.23606 +; 28, .37267 +; 29, 10.068 +.

Page 156. 3, $a - 3$; 4, $y + 4$; 5, $2a - 3b$; 6, $x^2 + 2x - 4$; 7, $x^2 - 4x + 2$; 8, $2x^2 - 3x + 1$; 9, $2x^2 + 4cx - 3c^2$; 10, $x^3 + 2x^2 - 4x - 8$; 11, $a + x + 5$; 12, $a - b + c$.

Page 158. 3, 24; 4, 426; 5, 75; 6, 4.78; 7, .27; 8, .899; 9, 3.27; 10, 4968; 11, $\frac{11}{17}$, $\frac{39}{41}$; 12, 1.259; 13, 1.442; 14, 1.709.

Page 161. 1, $x^{\frac{2}{3}}$, $a^{\frac{2}{3}}$; $a^{\frac{3}{4}}$, $a^{\frac{4}{5}}$, $a^{\frac{10}{9}}$, $a^{\frac{1}{2}}b^{\frac{1}{3}}$; 2, $x^{\frac{1}{2}}y^{\frac{3}{2}}$, $a^{\frac{3}{4}}b^{\frac{1}{2}}c^{\frac{1}{4}}$, $a^{\frac{3}{4}}b^{\frac{1}{2}}c^{\frac{1}{4}}$, $3mn^2x^{\frac{2}{3}}y^{\frac{1}{2}}$; 3, $\sqrt{a}$, $\sqrt[3]{a^3}$, $\sqrt[6]{a^3b}$, $4\sqrt[6]{x^2y^3z}$, $5\sqrt[12]{x^4y^3z^2}$; 4, $\frac{1}{a^3}$, $\frac{2}{x^2y^3}$, $\frac{5a}{b^4}$, $\frac{1}{x} + \frac{2y}{x^2} - \frac{3y^2}{x^3}$; 5, ab^{-1} , $abc^{-2}x^{-1}$, $mn^{-4}r^{-1}x^2$, $3^{-1}a^{-1}bc^{-1}$, $4^{-1}b^{-\frac{1}{2}}c^{\frac{1}{3}}$, $5a^2b^{-2}cd^{-1}xy^{-1}$; 6, $a^{\frac{5}{6}}$, $b^{\frac{11}{2}}$, $c^{\frac{19}{2}}$, $x^{\frac{22}{5}}$; 7, $a^{\frac{1}{6}}$, $n^{-\frac{1}{12}}$, $a^{-\frac{1}{2}}$, $m^{\frac{1}{2}}$.

Page 162. 8, a , $c^{\frac{1}{6}}$, $\sqrt[6]{c^{13}}$, $\sqrt[12]{x^{-7}}$; 9, $a^{\frac{3}{2}}b^{\frac{1}{6}}c^{\frac{4}{3}}$, $ab^{\frac{5}{6}}c^{\frac{1}{6}}$; 10, $a^{\frac{7}{12}}$, $b^{\frac{1}{6}}c^{-\frac{1}{4}}x^{\frac{1}{6}}$; 11, $a^{\frac{1}{6}}$, $c^{\frac{1}{20}}$, $x^{\frac{11}{10}}$, $y^{-\frac{5}{6}}$; 12, $n^{\frac{14}{5}}$, $x^{\frac{17}{2}}$, $2x^{\frac{1}{6}}y^{\frac{1}{6}}$; 13, $m^{\frac{8}{3}}$, $x^{-\frac{2}{3}}$, $y^{-\frac{1}{3}}$, z , $a^{-4}b^3$; 14, $\frac{1}{8}x^{-6}y^{-2}z^{\frac{3}{2}}$, $\sqrt[3]{\frac{1}{9}a^4b^{\frac{3}{2}}c^{-\frac{2}{5}}x^{-\frac{2}{3}}}$; 15, $729a^3b^4c^{\frac{9}{2}}$, $\sqrt[4]{-\frac{1}{8}a^{-1}x^{\frac{8}{3}}y^{-\frac{2}{5}}}$; 18, 8, $\frac{1}{8}$, $\frac{1}{32}$, $\frac{1}{\sqrt[8]{125}\sqrt[5]{a^2}}$.

Page 163. 20, $a^{-5} - 3a^{-\frac{3}{2}}b^2 + a^{-2}b^3 - 2a^{-1}b^4$; 21, $2x^{-1}y - 10xy^{-1} + 8x^3y^{-3}$; 22, $a^{\frac{3}{2}}b^{-\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{-\frac{1}{2}}b^{\frac{3}{2}}$; 23, $x^{\frac{5}{2}} + x^{\frac{3}{2}}y - xy^{\frac{3}{2}} - y^{\frac{5}{2}}$; 24, $a + a^{\frac{1}{2}}b + b^2$; 25, $6x^2 - 7x^{\frac{5}{3}} - 19x^{\frac{4}{3}} + 5x + 9x^{\frac{2}{3}} - 2x^{\frac{1}{3}}$; 26, $x^{-2} -$

$$3x^{-\frac{4}{3}} + 3x^{-\frac{2}{3}} - 1; 28, x + y; x^{\frac{3}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{3}{2}};$$

$$30, a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}; 31, x^{\frac{2}{3}} - x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}; 32, 2x^2 -$$

$$3xy + 2y^2; 33, 3x^{-2} - 3x^{-1}y^{\frac{1}{2}} + y; 34, 2a^{\frac{1}{2}} - 3b^{\frac{1}{2}} + 4c^{\frac{1}{2}}.$$

Page 165. Fill the blanks with: 3, $25a^2$; 4, $16a^6b^4c^2$; 6, $4x^2 + 4x + 1$; 7, $27c^6x^3$; 8, $x^3 - 6x^2 + 12x - 8$; 9, $81a^4x^8$; 10, $x^4 + 2x^3 + x^2$; 11, $\frac{8a^3}{27}$; 12, $-\frac{125a^6c^3}{216b^3}$.

Page 166. Fill the blanks with: 15, 180; 16, $(a + b)^2a$; 17, 243; 18, $(x^2 - 4)(x - 2)$; 20, c ; 21, $-24a^{11}b^7$; 22, $\frac{a}{c} + \frac{a^3}{c^3}$; 23, $\frac{8}{16}$; 24, $x^2 - a + \frac{a^2}{x^2}$; 26, 2; 27, 4; 28, 3; 29, 25; 31, 6; 33, 20; 35, $8a^2b^4c^3$; 38, $\frac{1}{14}$; 39, 2.

Page 167. 40, $3a^2\sqrt{b}$; 41, $10ab\sqrt{2a}$; 42, $14a^2b^2x\sqrt{2ax}$; 43, $15x^2\sqrt{3y}$; 44, $8a^2b^4c^3\sqrt{3ac}$; 45, $36a^2b^2c\sqrt{5ab}$; 46, $3a\sqrt[3]{3ab}$; 47, $2xy\sqrt[3]{20x^2}$; 48, $6x^2\sqrt[3]{4z}$; 49, $2a\sqrt[4]{5c}$; 50, $3ac\sqrt[4]{2ab^3c^3}$; 51, $8a^3b\sqrt[5]{2c}$; 52, $3xy\sqrt{2xy^2 - 3x^2y}$; 53, $(x - 3)\sqrt{a}$; 54, $(x + y)\sqrt{x - y}$; 55, $4ab\sqrt[3]{3ab^2 + 5b}$; 57, $\frac{1}{4}\sqrt{10}$; 58, $\frac{1}{8}\sqrt{30}$; 59, $\frac{1}{6}\sqrt{14}$; 60, $\frac{1}{15}\sqrt{165}$; 61, $\frac{1}{14}\sqrt{21}$; 62, $\frac{1}{4}\sqrt{6}$; 63, $\frac{1}{3}\sqrt{15}$; 64, $\frac{1}{b}\sqrt{ab}$; 65, $\frac{1}{2}\sqrt[3]{6}$; 66, $\frac{1}{5}\sqrt[3]{10}$; 67, $\frac{2}{11}\sqrt[3]{196}$; 68, $\frac{1}{b}\sqrt[3]{a^2b}$.

Art. 276. Fill the blanks with: 71, $8ab$; 72, $27a^6x^3$; 73, $2abc^3$; 74, $32a^5b^{10}x^{15}$; 75, $5abc^2$.

Page 168. Fill the blanks with: 79, a , a^6b^4 , $a^{12}b^{24}$; 80, x , ab^2 , $a^{\frac{2}{3}}b$.

Art. 277. 82 $\sqrt[12]{64}$, $\sqrt[12]{81}$, $\sqrt[12]{125}$; 83, $3^{\frac{2}{6}}$, $2^{\frac{3}{6}}$, $5^{\frac{1}{6}}$; 84, $\sqrt[6]{27}$, $\sqrt[6]{4}$, $\sqrt[6]{7}$; 85, $4^{\frac{9}{12}}$, $2^{\frac{8}{12}}$, $7^{\frac{6}{12}}$; 88, $\sqrt[10]{a^5}$, $\sqrt[10]{b^5}$, $\sqrt[10]{c^8}$; 89, $c^{\frac{8}{28}}$, $x^{\frac{21}{28}}$, $z^{\frac{2}{28}}$.

Page 169. 3, $3\sqrt{7}$; 4, $6\sqrt{5}$; 5, $6\sqrt{11}$; 7, $\sqrt[3]{2}$; 9, $\sqrt[3]{\frac{14}{15}}$; 10, $\sqrt[3]{10}$. Art. 279. 2, $7\sqrt{2}$; 3, 0; 4, $\sqrt{3}$; 5, $32\sqrt{2}$; 6, $8a\sqrt{c}$; 7, $4\sqrt[3]{5}$; 8, $7ab\sqrt{2a}$; 9, $6a^2\sqrt[3]{9c+20a^2\sqrt[3]{3c}}$; 10, $2\sqrt[3]{3}+3\sqrt[3]{2}$; 11, $11x^3\sqrt[3]{4y^2}$; 12, $2 \cdot x^3(a+1)\sqrt{3ay}$; 13, $a^2x\sqrt[3]{4x}$; 14, $\frac{1}{2}\sqrt[3]{6}$; 15, $\frac{1}{6}\sqrt{3}$; 16, $\frac{1}{4}\sqrt{6}$; 17, $\frac{adm-bc}{bdmn}\sqrt{mn}$; 18, $\sqrt[7]{\frac{3}{18}}$; 19, $\frac{a}{b}\sqrt[3]{4a^2b^2x}$.

Page 170. 3, $48a\sqrt{3c}$; 4, $3a\sqrt{3x}$; 5, $70ab\sqrt{2a}$; 6, $9a\sqrt{2}$; 7, $70a^3x\sqrt{2ac}$; 8, $12a^4\sqrt{3}$; 9, 108; 10, $\sqrt[4]{3}$; 11, $\sqrt[3]{\frac{1}{14}}$; 12, 1; 13, $4\sqrt[3]{15}$; 14, $\sqrt[2]{\frac{3}{36}}$; 15, $12\sqrt[6]{432}$; 16, $\sqrt[3]{\frac{6}{72}}$; 17, $\sqrt[12]{\frac{1}{12}}$; 18, $\sqrt[10]{\frac{81}{1024}}$.

Page 171. 20, 2; 21, $14+8\sqrt{3}$; 22, $-22-7\sqrt{6}$; 23, $-39+7\sqrt{7}$; 24, $4+2\sqrt{10}$; 25, $56+12\sqrt{35}$; 26, $a+c-b+2\sqrt{ac}$; 27, $11x-9-3\sqrt{1-x^2}$; 28, $-78-47x+17\sqrt{6+13x+6x^2}$; 29, $20x-12+12\sqrt{1-x-6a^2}$.

Art. 281. 3, $16a^2$; 4, $125a^3x$; 5, $9\sqrt[3]{m}$; 6, $96x^8\sqrt[3]{6}$; 7, $-\sqrt[3]{a^2x}$; 8, $-a^7x^9\sqrt{a}$; 9, $\frac{5}{27}$; 10, $\frac{1}{32}$; 11, $x^2\sqrt[3]{y}$; 12, $\sqrt[3]{y}$.

Page 172. 13, $3 + 2\sqrt{2}$; 14, $10 - 6\sqrt{3}$; 15, $38 + 17\sqrt{5}$; 16, $11 + 6\sqrt{2}$; 17, $56 - 24\sqrt{5}$; 18, $9\sqrt{3} + 11\sqrt{2}$; 19, $4 + 4\sqrt[3]{3} + \sqrt[3]{9}$; 20, $25 - 27\sqrt[3]{2} + 9\sqrt[3]{4}$; 21, $2\sqrt{2} + 6\sqrt[3]{4} + 6\sqrt{2}\sqrt[3]{2} + 4$; 22, $a + 2\sqrt[6]{a^3b^2} + \sqrt[3]{b^2}$; 23, $a^2 + 3a\sqrt{a} + 3a + \sqrt{a}$; 24, $a^2 + 4a\sqrt[6]{a^3b^2} + 6a\sqrt[3]{b^2} + 4b\sqrt{a} + b\sqrt[3]{b}$; 25, $2 + 3x + x^2 + 2(1+x)\sqrt{1+x}$; 26, $2 + 2\sqrt{1-x^2}$.

3, $2\sqrt{a}$; 4, $\sqrt{5}\sqrt[3]{4}$; 5, $\pm\sqrt{3}\sqrt[6]{a}$; 6, $3\sqrt[4]{2}$; 7, $x^2\sqrt[4]{a}$; 8, $\sqrt{ab}$; 9, $\sqrt{\frac{a}{3}}$; 10, $\sqrt[3]{\frac{4}{3}}$; 11, $\sqrt{x} + \sqrt{y}$; 12, $\sqrt[4]{a} - \sqrt[4]{b}$; 13, $\sqrt[6]{a} + \sqrt[6]{b}$; 14, $2\sqrt[3]{a} - 5\sqrt[3]{b}$; 15, $\sqrt[3]{a} - \sqrt[3]{x}$; 16, $\sqrt{a} + \sqrt{b} - \sqrt{c}$.

Page 173. 19, $1 + \sqrt{3}$; 20, $2 - \sqrt{3}$; 21, $\sqrt{2} + \sqrt{5}$; 22, $2\sqrt{2} + \sqrt{10}$; 23, $\sqrt{5} + 2\sqrt{3}$; 24, $\sqrt{10} - \sqrt{3}$; 25, $\sqrt{3} + \sqrt{7}$; 26, $3 - \sqrt{2}$; 27, $2\sqrt{5} - 3\sqrt{2}$; 28, $\sqrt{11} - \sqrt{5}$; 29, $5 + \sqrt{10}$; 30, $\sqrt{21} - \sqrt{3}$.

Page 175. 3, $-3\sqrt{2}$; 4, 2; 5, $4\sqrt{-6}$; 6, $\sqrt{-3}$; 7, -24; 8, 2; 9, $-64\sqrt{-1}$; 10, -75; 11, 324; 12, $128\sqrt{-2}$; 13, 29; 14, 107; 15, 83.

Page 176. 16, -43; 17, $-11 - 2\sqrt{-1}$; 18, -8; 19, $-6\sqrt{-1}$. 2, $\frac{3}{2}\sqrt{2}$; 3, $\frac{5}{4}\sqrt{2}$; 4, $\sqrt{2}$; 5, $\frac{2\sqrt{3}+3}{3}$;

$$6, \frac{2 - \sqrt[3]{10}}{4}; 7, -\frac{3\sqrt{-2} + \sqrt{-10}}{2}.$$

Page 177. 8, $\frac{\sqrt[4]{2} + \sqrt[4]{10}}{2}$; 9, $2\sqrt[3]{3} + \sqrt[3]{72}$. 10, $n\sqrt[5]{a} + \sqrt[5]{a}\sqrt{-2}$; 12, $\frac{19 - 8\sqrt{3}}{13}$; 13, $\frac{7 + 3\sqrt{5}}{2}$; 14, $-5 - 2\sqrt{6}$; 15, $\frac{18 + 10\sqrt{-7}}{32}$; 16, $-\frac{1 + 2\sqrt{-2}}{3}$; 17, $-\frac{7 + 6\sqrt{-2}}{11}$; 18, $-\frac{5 + a + 4\sqrt{a+1}}{a}$; 19, $2a^2 - 1 - 2a\sqrt{a^2 - 1}$; 20, $\frac{x^2 - 2 + x\sqrt{x^2 - 4}}{2}$; 21, $\frac{a + \sqrt{a^2 - x^2}}{x}$; 22, $\sqrt{a^4 - 1} - a^2$.

Page 180. 2, $x = 3, y = 4$; 3, $x = 4\frac{1}{2}, y = 6$; 4, $x = 4, y = 4$; 5, $x = 9, y = 12$; 6, $x = 3, y = 2$; 7, $x = 7, y = 5$.

Page 181. 4, 3; 5, 5; 6, 5; 7, $-\frac{6}{11}$; 8, 9; 9, 4; 10, 4; 11, 12; 12, 81; 13, 5; 14, 2; 15, 4; 16, 25; 17, 0; 18, 4; 19, 0.

Page 182. 20, 4; 21, 6; 22, $\frac{a - c^2}{2c}$; 23, $c(\sqrt{n} - a)$; 24, 25; 25, $56\frac{1}{4}$; 26, 6; 27, $\frac{9b^2}{a}$; 28, 6; 29, $\frac{4}{9}$; 30, $\frac{b^2 - 4a^2}{4a}$.

Page 184. 3, ± 8 ; 4, ± 7 ; 5, ± 6 ; 6, $\pm \sqrt{11}$; 7, ± 4 ; 8, ± 9 ; 9, $\pm \frac{1}{2}$; 10, $\pm \frac{7}{3}$; 11, ± 1 ; 12, $\pm \frac{1}{2}$; 13, $\pm \sqrt{11}$; 14, $\pm \frac{3}{2}\sqrt{-2}$; 15, $+1$; 16, $\pm \frac{3}{4}$; 17, $\pm \frac{24a}{7}$; 18, $\pm \frac{a(b^2 - c^2)}{2bc}$.

Page 185. 4, — 2, — 9; 5, — 3, — 7; 6, 3, 12; 7, 2, 11; 8, 3, — 8; 9, 4, — 11; 10, 8, — 2; 11, 9, — 4; 12, — 6, — 6; 13, 7, — 12; 14, $\frac{2}{3}$, — $\frac{1}{4}$; 15, $\frac{2}{3}$, — 2; 16, — 3, — $\frac{1}{4}$; 17, $\frac{1}{4}$, — 3.

Page 186. 21, + 2, — 8; 22, + 1, — 9; 23, + 10, — 2; 24, + 12, — 2; 25, + 1, — 13; 26, $7 + \sqrt{170}$, $7 - \sqrt{170}$; 27, 1, — 4; 28, 2, — 7.

Page 187. 29, 10, — 3; 30, 20, — 5; 31, — 1, — 9; 32, — 1, — 17; 33, 2, 20; 34, $\frac{1}{2}(11 + \sqrt{13})$, $\frac{1}{2}(11 - \sqrt{13})$; 35, — 6, — 7; 36, 6, 13; 37, 7, — $\frac{1}{3}$; 38, 4, — $\frac{3}{2}$; 39, 5, — $6\frac{1}{2}$; 40, 5, — $5\frac{2}{3}$; 41, 3, $3\frac{1}{6}$; 42, 9, — $8\frac{2}{7}$; 43, 5, — $5\frac{3}{4}$; 44, $\frac{37}{3}$, $\frac{13}{4}$; 45, 4, — $\frac{1}{4}$; 46, 24, — 6; 47, 4, — $\frac{5}{3}$; 48, 3, $\frac{1}{2}$; 49, $7\frac{1}{3}$, 4; 50, 8, 2; 51, b , — $b - 2a$; 52, $3n$, $4c - 3n$; 53, $\frac{a}{b}$, $\frac{b}{a}$; 54, a , — $\frac{b}{a}$.

Page 188. 58, 3, — $\frac{1}{3}$; 59, $\frac{17}{7}$, — 3; 60, $\frac{7}{6}$, — 3; 61, — $\frac{1}{3}$, — 8; 62, 9, $\frac{1}{5}$; 63, $\frac{3}{2}$, $\frac{4}{5}$; 64, — $\frac{1}{6}$, — $\frac{1}{2}$; 65, $\frac{1 \pm \sqrt{409}}{8}$; 66, 7, $\frac{5}{8}$; 67, 4, — $\frac{7}{4}$; 68, 4, — $1\frac{2}{3}$; 69, — 3, $2\frac{7}{22}$; 70, 10, — $\frac{2}{3}$; 71, 7, — $\frac{19}{7}$; 72, 9, $1\frac{5}{3}$; 73, 6, $4\frac{9}{3}$; 74, $\sqrt{x} = -5$; 75, 8, — $\frac{8}{3}$.

Page 189. 76, $\frac{7}{4}$, 1; 77, $\frac{13}{3}$, 1; 78, 0, 4; 79, 1, — $\frac{3}{4}$; 80, $a \pm b$; 81, $\frac{b}{a}$, — $\frac{d}{c}$; 82, 0, $\frac{2ab - ac - bc}{a + b - 2c}$; 83, $\frac{ab + bc + ac \pm \sqrt{a^2b^2 + b^2c^2 + a^2c^2 - abc(a + b + c)}}{a + b + c}$; 84, $\frac{2a}{a + b}$, — $\frac{2b}{a + b}$; 85, $\frac{p\sqrt{q} \pm q\sqrt{p}}{p - q}$; 86, $\frac{n}{n^2 - m^2}(bn \pm \sqrt{a^2m^2 + b^2m^2 - a^2n^2})$.

Page 190. 4, $x^2 - 14x + 45 = 0$; 5, $x^2 - 2ax + a^2 - b^2 = 0$; 6, $x^2 - 5x - 84 = 0$; 7, $x^2 - 2bx - a^2 + b^2 = 0$; 8, $x^2 + 7x - 30 = 0$; 9, $x^2 - 6c + 9c^2 - 1 = 0$; 10, $x^2 + 12x + 35\frac{3}{4} = 0$; 11, $x^2 - 6x + 7 = 0$; 12, $x^2 + \frac{b^2 - a^2}{ab}x - 1 = 0$; 13, $x^2 - ax + \frac{a^2 - b^2}{4} = 0$; 14, $x^2 - \frac{ax}{12} - \frac{a^2}{2} = 0$; 15, $x^2 - ax + \frac{a^2 + b^2}{4} = 0$; 16, $x^2 - ax - b = 0$; 17, $ax^2 + bx + c = 0$.

NOTE. In quadratic equations all the answers to each will not always be given.

Page 196. 7, $\pm 2, \pm \sqrt{10}$; 8, 4, $\frac{1}{4}$; 9, 1, -2; 10, 8, $\frac{125}{64}$; 11, 9, $(-41)^{\frac{2}{3}}$; 12, 1; 13, 16, $\frac{14641}{625}$; 14, 4, -9; 15, 1, $\frac{7}{9}\sqrt{63}$; 16, 0, -5, $\frac{1}{3}$, $-\frac{16}{3}$; 17, $\frac{1}{4}$, $\frac{1}{9}$; 18, 1, -4, $\frac{-3 \pm \sqrt{109}}{2}$; 19, 1, $\frac{1}{9}$; 20, $-1 \pm \sqrt{3}$, $-1 \pm \sqrt{2}$; 21, 2, -2, 3, 7; 22, 3, $\frac{1}{3}$, $\frac{5 \pm \sqrt{28}}{3}$.

Page 197. 23, 1, -1, 5, 7; 24, 4, 1, 3, 2; 25, -1, -2, $\frac{-3 \pm \sqrt{33}}{2}$; 26, $\frac{-3 \pm \sqrt{29}}{4}$, $\frac{-3 \pm \sqrt{-7}}{4}$.
3, 3, 17; 4, 7, 13; 5, $10 + \sqrt{-5}$, $10 - \sqrt{-5}$.

Page 198. 7, 3; 8, 4; 9, $\frac{11 \pm \sqrt{-23}}{2}$; 12, \$4.
13, 4 mi.

Page 199. 14, 10, 11, 15, 14, 15; 16, 6, 7, 8, 9; 17, 14 rd., 16 rd.; 18, 7 ch., 14 ch.; 19, 3 ft.; 20, 10 in.; 21, 12, 18; 22, 3, -10; 23, $\frac{15}{20}$.

Page 200. 24, 6 hr.; 8 hr.; 25, 40 ft.; 26, 6, 18; 27, 2 ft., 5 ft.; 28, 30s. per week; 29, A \$200, B \$300; 30, 76 mi.

Page 201. 2, $(900)^2$ acres; 3, $x = 3, y = 4$; 4, $x = 4, y = 3$; 5, $x = 7, y = 4$; 6, $x = 3, \frac{5}{2}\sqrt{2}, y = 2, \frac{1}{2}\sqrt{2}$; 7, $x = \frac{5}{7}\sqrt{21}, y = \frac{1}{7}\sqrt{21}$; 8, $x = 3, y = -4$.

Page 203. 12, $x = 2, 5, y = 5, 2$; 13, $x = 8, -2, y = 2, -8$; 14, $x = 7, 11, y = 11, 7$; 15, $x = 12, -7, y = 7, -12$; 16, $x = 8, 6, y = 6, 8$; 17, $x = 15, -10, y = 10, -15$; 18, $x = 24, 10, y = 10, 24$; 19, $x = 12, -5, y = 5, -12$; 20, $x = 2, 3, y = 3, 2$; 21, $x = 2, 5, y = 5, 2$; 22, $x = \frac{1}{5}, \frac{1}{4}, y = \frac{1}{4}, \frac{1}{5}$; 23, $x = \frac{1}{7}, -\frac{1}{6}, y = \frac{1}{6}, -\frac{1}{7}$; 24, $x = \pm 2, y = \pm 1$; 25, $x = 3, y = 2$; 26, $x = 5, y = 2$; 27, $x = 3, y = 4$.

Page 204. 30, $x = 7, y = 4$; 31, $x = 10, y = 3$; 32, $x = 8, y = 5$; 33, $x = 2, y = 1$; 34, $x = \pm 3, \pm 2, y = \pm 2, \pm 3$; 35, $x = \pm 5, \pm 3, y = \pm 3, \pm 5$; 36, $x = 8, 2, y = 2, 8$; 37, $x = 9, 4, y = 4, 9$; 38, $x = 3, y = 2$; 39, $x = 9, y = 4$; 40, $x = 8, y = 1$; 41, $x = 3, y = 2$.

Page 205. 43, $x = 3, 36, y = 5, -\frac{2}{3}$; 44, $x = 2, \sqrt{\frac{2}{5}}, y = \frac{1}{2}, -2\sqrt{\frac{2}{5}}$; 45, $x = 2, -2, \frac{5}{\sqrt{31}}, y = 3, -3, -\frac{6}{\sqrt{31}}$; 46, $x \pm 1, \pm 14\frac{3}{4}, y = \mp 3, \pm 3\frac{7}{8}$.

Page 206. 2, 23, 17; 3, 13, 6; 4, 9, 7; 5, 3, 2; 7, 23; 8, 5, 6, 7, 8, or 1, 2, 3, 4.

Page 207. 10, A 6 mi., B 4 mi.

Page 208. 11, 30 rd., 20 rd.; 12, 14; 13, 18; 15, 9, 6, 4.

Page 209. 17, \$25, \$16; 18, 50 ct., 25 ct; 19, 20 da., 30 da.; 20, 18 ft., 16 ft.

Page 210. 21, A 40 hr., B 60 hr.; 22, 36 men, 77 lb., or 28 men, 45 lb.; 23, 42 ft; 24, 48 ft.

Page 211. 25, 5 mi., 2 mi. per hour; 26, 9 ct.; 27 \$2, \$3, and 60, 40 sheep; 28, street 73 ft., ladder 53 ft.; 29, $\frac{1}{2}\sqrt{5}$, $\frac{1}{4}(5 + \sqrt{5})$.

Page 216. 2, $\frac{577}{408}$; 3, $\frac{5201}{1020}$; 4, $\frac{721}{228}$.

Page 217. 2, $\frac{6581}{4563}$; 3, $\frac{375047}{297675}$; 4, $2.223 +$

Page 224. 3, 5 : 6; 4, $\frac{1}{20}$; 5, $\frac{2}{35}$; 6, 24; 7, 275; 8, 1, 9; 9, 2, — 7; 10, $4\frac{1}{2}$; 11, 12, 16; 12, 50, 60; 13, A 27, B 63; 14, $\frac{55}{21}$; 15, $3.14 +$; 16, 8, 12.

Page 225. 17, short road from A to B 26 mi., from B to C 52 mi.

Page 229. 8, 12; 9, 35; 10, $4\frac{2}{5}$; 11, 13; 12, $\frac{4}{7}$, 3; 13, $\frac{c(a+c)^2}{(a-c)^2}$; 14, ± 1 ; 15, 18, 27; 16, 18, 14; 17, 15yd., 25yd.; 18, 6, 9; 19, $5\frac{1}{10}$, $4\frac{5}{10}$; 20, 28, 21.

Page 230. 23, $\$25\frac{3}{8}$; 24, 2, 4; 25, $171\frac{1}{2}$ lb.; 26, 3, 6.

Page 233. 7, $y = \frac{3}{2}x$; 8, 20.

Page 234. 9, $z = 3(x + y)$; 10, $y = \sqrt[3]{\frac{x^2}{2}}$; 11, $z = 3xy$; 13, $u \propto xyz$; 14, 16.

Page 235. 17, $\$26\frac{1}{4}$; 18, \$18; 19, $\frac{mr}{n}$ ft.; 20, $3\frac{3}{5}$ da.; 21, 20.

Page 236. 23, 495 lb.; 24, 30; 26, 55 hr.

Page 238. 1, 149, 2, 64; 3, 5; 4, — 11; 5, 42; 6, 71; 7, 53.

Page 239. 8, 670; 9, 884; 10, 500500; 11, $\frac{1}{2}n(n+1)$; 12, 10201; 13, n^2 ; 14, 1092.

Page 240. 15, 800; 16, 4; 17, $61\frac{1}{2}$; 18, 425; 19, $n(8+n)$; 20, $\frac{n(13-n)}{12}$; 21, 2235; 22, $\frac{n}{12}(13-7n)$; 23, $\frac{n-1}{2}$; 25, 9; 26, 6, 10, 14, 18, 22; 27, $8\frac{1}{2}$, 12, etc., to $29\frac{1}{2}$; 28, 3.1, 3.2, etc., ... 3.9; 29, $4\frac{1}{4}$, $4\frac{1}{8}$, $4\frac{5}{12}$, etc., to $5\frac{5}{12}$.

Page 241. 31, 5; 32, 15; 33, 5; 34, — 13; 35, 11; 36, $\frac{1}{2}$.

Page 242. 39, 7; 40, $l=7$, $d=-\frac{1}{3}$; 41, $a=2$, $n=11$; 42, 5.4, 5.8, 6.2, 6.6; 43, 1, 3, 5, etc.; 44, 0, 1, 3, 6, 10, etc.

Page 243. 45, $a=\frac{1}{3}$, $d=\frac{1}{2}$, $s=\frac{n}{12}(3n+1)$; 46, 20; 47, 4 or 7; 48, $-n(-1)^n$; 49, $\frac{1}{4}[1-(2n+1)(-1)^n]$; 50, 3, 5, 7, 9; 51, 5, 7, 9, 11; 52, 8, 10, 12; 53, 2, 3, 4; 54, 5, 6, 7, 8, 55, 2, 3, etc. to 8; 56, 5 hr.

Page 244. 57, 3, 8, 13, etc.; 58, 3 or 10 da. and 36 or 120 mi.; 59, $948\frac{1}{2}$ and 14475 ft.; 60, 19 mi. 160 rd. 640 ft.

Page 245. 2, 327680; 3, $1\frac{3}{4}$; 4, 320.

Page 246. 5, $\frac{4}{3}$; 6, $8\frac{2}{3}$; 7, 2047; 8, $131071\frac{1}{2}$; 9, 9841; 10, $3249\frac{7}{8}$; 11, $\frac{1}{5}[(\frac{5}{3})^6-1]$; 12, $-\frac{2}{3}(2^{10}-1)$; 13, $9[1-(\frac{2}{3})^n]$; 14, $\frac{3^n-1}{2^{n-1}}-\frac{2}{3}$.

Page 247. 16, $13\frac{1}{2}$; 17, $1\frac{1}{3}$; 18, $\frac{59}{11}$; 19, 2; 20, $\frac{27}{6}$;
21, $\frac{4}{3}$; 22, $\frac{4}{8}$; 23, $\frac{n}{n-1}$; 24, $\frac{n}{n+1}$.

Page 248. 26, 6; 27, $\frac{1}{3}$; 28, $\sqrt{-1}$; 29, 20. 80;
30, 18, 54, 162; 31, 6, 18, 54, etc.; 32, $(-1)^{\frac{1}{2}}$,
 $(-1)^{\frac{1}{3}}$, $(-1)^{\frac{3}{4}}$; 33, $(-1)^{\frac{1}{6}}$, $(-1)^{\frac{1}{3}}$, $(-1)^{\frac{1}{2}}$, etc.

Page 249. 34, 3; 35, $r = 5$, $s = 15624$; 36, 2;
37, 5; 38, $n = 9$; 39, $\frac{3}{8}$, $\frac{9}{32}$; 40, 32; 41, 1, 2, 4; 42,
7, 14, 28, 56.

Page 250. 43, 1, 3, 9, 27; 44, 1, 3, 9; 45, 2, 4, 8,
16; 46, 6, 8, 10, 12; 47, 400 mi.; 48, \$21, \$42, \$84;
49, 9, 27, 81, etc.; 50, £3 4s; 51, 2, 4, 8, 12.

Page 252. 7, 4, 6; 8, $\frac{1}{3}\frac{5}{1}$.

Page 253. 9, 7, -21 , $-\frac{21}{5}$, $-\frac{7}{3}$, $-\frac{21}{13}$, $-\frac{21}{17}$;
10, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$; 11, $\frac{6}{11}$, $\frac{6}{14}$; 12, 2, 8.

